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ABSTRACT

This document represents the next step by the National Council of Teachers of Mathematics (NCTM) in its efforts to ensure quality, indicate goals, and promote positive change in mathematics education for grades pre-K-12. Following an introduction in Chapter 1, Chapter 2 elaborates on six guiding principles for school mathematics instructional programs. An overview of the 10 standards is provided in Chapter 3, followed by grade band elaboration chapters for grades pre-K-2 (Chapter 4), grades 3-5 (Chapter 5), grades 6-8 (Chapter 6), and grades 9-12 (Chapter 7). Chapter 8 is planned for inclusion in the final version of Principles and Standards that will focus on implementation and policy implications. This draft offers an outline of Chapter 8 and invites readers to submit comments and reactions. This draft of Principles and Standards includes a number of substantive and structural features that are different from the original Standards documents, the most significant structural feature being that this product is presented in both print and electronic format. The production of this document in electronic format will allow a richer array of examples to illuminate and extend the ideas provided in the text and make possible the inclusion of links to resource and background materials. Contains 160 references. (ASK)

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NCTM

Principles and Standards for School Mathematics:

Discussion Draft

October 1998

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STANDARDS 2000

NCTM
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

SE062993

National Council of Teachers of Mathematics
1906 Association Drive
Reston, VA 20191

October 15, 1998

Dear Colleague:

This is an invitation to read, reflect and react to this discussion draft of NCTM's updated *Standards* entitled *Principles and Standards for School Mathematics*. Please take this opportunity to help shape the final version of this important effort of the Council.

We have had a decade of experience with the previous NCTM *Standards*. Many of you have been involved in local, state, and provincial efforts to use the *Standards* to improve mathematics teaching and learning. The previous *Standards* guided the development of many curriculum guides, state and provincial standards, local curriculum and assessment materials and professional development programs. Much has been learned from these efforts—from continued research on mathematics teaching and learning, and from continued dialogue on the previous *Standards*.

NCTM has produced this discussion draft of *Principles and Standards for School Mathematics* as part of the process of supporting the development of a high quality mathematics program in every classroom. Now we need your help. We need your comments and suggestions. We need your examples. We want to hear your views and reasons—both agreements and disagreements.

As you read the draft, you will notice interspersed "Reader Reaction" boxes. We need your best advice on these questions, as well as the question in the "Shaping the Standards" features in the NCTM journals and questions included in the electronic version. However, the best questions come from you. We need the insights you have gained and your reactions to this draft to help make the final version of *Principles and Standards for School Mathematics* as helpful as possible to as many as possible.

The following page gives information on how you can submit your reactions. We look forward to hearing from you prior to May 1, 1999.

Sincerely,

Glenda Lappan
President
National Council of
Teachers of Mathematics

Joan Ferrini-Mundy
Chair
Writing Group

Mary M. Lindquist
Chair
Commission on the Future
of the Standards

RESPONDING TO THE DRAFT

We would like your input in shaping the *Principles and Standards for School Mathematics* so that the final version is a usable and useful document.

There are several ways in which you can give your reaction:

Fill out a form on the Standards 2000 Web site—*PREFERRED!*
http://www.nctm.org/standards2000/

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Please specify each question to which you respond. Attach extra sheets as necessary.

NCTM

Principles and Standards for School Mathematics:

Discussion Draft

October 1998

Prepared by the Standards 2000 Writing Group
with support from
the Commission on the Future of the Standards
and the Electronic Format Group

This Discussion draft is a working document
and does not represent official policy of
the National Council of Teachers of Mathematics.
Comments and reactions are welcome.

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STANDARDS 2000 PROJECT

Writing Group

Joan Ferrini-Mundy, *Chair*, National Research Council and Dept. of Mathematics, University of New Hampshire

W. Gary Martin, *Staff Liaison*, National Council of Teachers of Mathematics

Grades Pre-K–2

Jeane Joyner (chair), North Carolina Dept. of Public Instruction

Angela Andrews, Scott School, Naperville, IL

Douglas Clements, Dept. of Learning and Instruction, State University of New York at Buffalo

Alfinio Flores, Division of Curriculum and Instruction, Arizona State University

Carol Midgett, Southport Elementary School, Southport, NC

Judith Roitman, Dept. of Mathematics, University of Kansas at Lawrence

Grades 3–5

Barbara Reys (chair), Dept. of Curriculum and Instruction, University of Missouri at Columbia

Frances (Skip) Fennell, Education Dept., Western Maryland College at Westminster

Cathy Fueglein, Webster Groves School District, Webster Groves, MO

Melinda Hamilton, Rosemont Elementary School, Orlando, FL

Susan Jo Russell, TERC, Cambridge, MA

Philip Wagreich, Dept. of Mathematics, Statistics and Computer Science, University of Illinois at Chicago

Grades 6–8

Ed Silver (chair), Learning Research and Development Center, University of Pittsburgh

Mary Bouck, Battle Creek Public Schools, MI

Jean Howard, C. R. Anderson Middle School, Helena, MT

Diana Lambdin, School of Education, Indiana University at Bloomington

Carol Malloy, School of Education, University of North Carolina at Chapel Hill

James Sandefur, Department of Mathematics, Georgetown University,Washington, DC

Grades 9-12

Alan Schoenfeld (chair), Dept. of Education, University of California at Berkeley

Sue Eddins, Illinois Mathematics and Science Academy, Aurora

Kathleen Heid, Pennsylvania State University

Millie Johnson, Dept. of Mathematics, Western Washington University at Bellingham

Ron Lancaster, St. Mildred's Lightbourn School, Ontario

Alfred Manaster, Dept. of Mathematics, University of California at San Diego

Milton Norman, Granby High School, Norfolk, VA

Electronic Format Group

Enrique Galindo (chair), Department of Curriculum & Instruction, Indiana University

Tom Gorski, Gilman School, Baltimore, MD

Beverly Hunter, Learning Communities Research Group, Boston College School of Education

Eugene Klotz, Swarthmore College (PA)

Nanette Seago, Learning About Teaching Mathematics: Using Digital Videocases Project (Riverside, CA)

Len Simutis, Eisenhower National Clearinghouse

Past Participants

Ed Esty, Staff Liaison

Jim Leitzel (deceased), Commission on Future of the Standards

Melissa Manzano, Grades 3–5 Writing Group

Linda Rosen, Commission on the Future of the Standards

Jim Rubillo, Commission on the Future of the Standards

Commission on the Future of the Standards

Mary Lindquist (chair), Columbus State University (Georgia)

Fred Crouse, Annapolis Valley Regional School Board (Nova Scotia, Canada)

Portia Elliott, University of Massachusetts

Mazie Jenkins, Madison (WI) school system

Jeremy Kilpatrick, University of Georgia

Michael Koehler, Overland Park (KS) school system

Marilyn Mays, North Lake College (Texas)

Richard Schoen, Stanford University

Bonnie Walker, city, TX

Glenda Lappan (ex officio), NCTM President

Gail Burrill (ex officio), NCTM Past President

John Thorpe (ex officio), NCTM Executive Director

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Preface

In its introduction to the *Curriculum and Evaluation Standards for School Mathematics* (1989), the National Council of Teachers of Mathematics (NCTM) offered three reasons for a professional organization to formally adopt standards: to ensure quality, to indicate goals, and to promote change. Periodic review and reformulation of standards affords those who are concerned with the improvement of education an opportunity to consolidate what is known from experience and to progress toward the goal of high-quality education for all students. This draft document, *Principles and Standards for School Mathematics: Discussion Draft* (hereafter called *Principles and Standards*), represents a next step by NCTM in efforts to ensure quality, indicate goals, and promote positive change in mathematics education in grades pre-K – 12.

The process of development for *Principles and Standards for School Mathematics: Discussion Draft* has been an elaborate one. The Writing Group first met in the summer of 1997 to begin to frame the project and discuss the issues involved. In the 1997-98 academic year, the Commission on the Future of the Standards orchestrated a coordinated process of gathering input from a range of communities, including the NCTM membership and the Association Review Groups, representing professional organizations in mathematics and mathematics education. Two special conferences were held to inform the process: the Conference on Foundations for School Mathematics in Atlanta in March 1998, organized by Jeremy Kilpatrick and sponsored by the National Science Foundation, and the Technology and NCTM Standards 2000 Conference, organized by Enrique Galindo and sponsored by the Eisenhower National Clearinghouse, in Washington, DC in June 1998. A set of white papers was commissioned by NCTM's Research Advisory Committee to assist the authors with the development of ideas in several of the content and process areas. In addition, collections of curriculum materials, state and province curriculum documents, research publications, policy documents, and international frameworks and curriculum materials were made available. All of the input was made available to the Writing Group, and we benefited greatly from its availability.

Principles and Standards is intended for discussion and reaction. The 1998-99 academic year will serve as a period for all who are interested in mathematics education to formulate their response, critique, and encouragement of elements of this document. Boxes in the text invite readers' comments, suggestions, input, vignettes, and examples. In the summer of 1999 the Writing Group will develop a final draft based on the input and commentary produced in the next several months. The positions described herein are not official NCTM positions.

Standards and the Nature of School Mathematics

The trio of documents produced by NCTM in the past decade—the *Curriculum and Evaluation Standards for School Mathematics* in 1989, the *Professional Standards for Teaching Mathematics* in 1991, and the *Assessment Standards for School Mathematics* in 1995 (hereafter called the *Standards*)—challenged the assumption that mathematics is only for the select few with a persuasive argument that everyone needs to understand mathematics and that there should be no conflict between equity and excellence. The original *Standards* articulated the view that all students should gain mathematical power—that is, the ability to explore, conjecture, and reason logically as well as the ability to use a variety of mathematical methods effectively to solve nonroutine problems (NCTM 1989).

With standards for problem solving, communicating, reasoning, and making connections, the 1989 *Curriculum and Evaluation Standards* portrays mathematics as something that is *done*, not merely a body of material to be memorized. By calling for the inclusion of mathematical content not widely addressed in the school curriculum, such as statistics, probability, and discrete mathematics, the 1989 *Standards* acknowledges the need for school mathematics to respond to and anticipate the changing needs of an increasingly technological and quantitative world.

The original NCTM *Standards* documents made it clear that a range of pedagogical strategies is necessary to support students' mathematics learning; that students learn to reason and communicate mathematically only if they have the opportunity to do so; and that individual, small-group, and whole-class instruction are all necessary components of a well-orchestrated mathematics learning environment. The complex relationships among problem solving, conceptual understanding, and procedural facility also are addressed in the original *Standards*. Finally, these documents made it clear that assessment at all levels—assessment of individual students, of the instructional environment, and of all aspects of the instructional system—is an integral part of the teaching-learning process and not merely a summative "add on" at the end.

NCTM's development of curriculum, teaching, and assessment standards represented an historically important first attempt by a professional organization to provide explicit and extensive goals statements for teachers. Although there will never be complete consensus within the mathematics education profession or among the general public about the ideas advanced in any standards documents, what is important is that standards provide a vehicle for focused, sustained efforts toward the improvement of students' mathematical education in school. In particular, standards have the capacity to offer language, examples, and recommendations that enable involvement by many groups of people.

Revision of the original NCTM *Standards* documents was planned in the early development of the 1989 *Curriculum and Evaluation Standards*. NCTM leaders recognized that whatever vision they provided for school mathematics would need examination and testing by practitioners and that changes of various types, such as new knowledge from research and new technologies, would inevitably necessitate reflection and reconsideration of the *Standards* ideas and messages.

This draft of *Principles and Standards* is informed substantially by the field's ten years of experience with considering and implementing standards for mathematics education. Interpretation of the original *Standards*, for example, has occurred widely in a variety of forms. Some interpretation has been provided by NCTM, through materials such as the *Addenda Series*. (Del Grande 1998; Burton 1991a, 1991b, 1991c, 1992a, 1992b, 1992c, 1993; Coburn 1992; Curcio 1994; Froelich 1991; Lindquist 1992; Meiring 1992; Phillips 1991; Reys 1991; Zawojewski 1991). Some interpretation has occurred in the field, as in state and provincial frameworks, curriculum programs and textbooks, and classroom instruction. As teachers and policymakers have worked with the ideas of the *Standards*, everyone has learned where more detail is needed, where more clarity is essential, and where messages need to be stronger or more refined.

Issues about the fidelity of interpretations to the original *Standards* documents are complex. The original *Standards* intended to provide guidance and vision without being overly prescriptive and left specific curricular decisions to the local level. As a result, an enormous range of policies and practices claim to fit beneath the *Standards* umbrella. Another result is that the *Standards* may be implemented in curricula or instructional practice in ways unintended by the original *Standards* writers. Examination of such implementations may therefore lead to inappropriate claims about the effects—both positive and negative—of the *Standards* in situations that are far removed from the original documents. And the original *Standards'* emphasis on pedagogical directions, such as discourse and the use of tools to enhance discourse (NCTM 1991, pp. 34–54), has created challenges for teachers. Incorporating such directions into teaching practice while at the same time maintaining focus on substantial mathematical learning is difficult.

With the hindsight of ten years of experience, it is clear that certain messages of the original *Standards* documents were not understood as they were intended. For instance, the charts in the *Curriculum and Evaluation Standards* that listed "Topics to Receive Decreased Attention" (NCTM 1989, 21, 71, 73, 127) were sometimes interpreted as advocating the complete removal of certain topics from the curriculum. As teachers and policymakers have worked with the *Standards* documents in the past decade, it also has become clear that certain ideas in the *Standards*—such as the role of basic skills and conceptual learning—needed refinement and elaboration. In producing this draft of *Principles and Standards,* we have benefited from recognizing the difficulties surrounding issues of interpretation.

An Overview of Principles and Standards

The charge to the *Principles and Standards* Writing Group was to "build on the foundations of the original NCTM *Standards* documents and to consolidate the classroom aspects of all three documents." In addition, the Writing Group was charged to organize this document into four grade band sections, thereby allowing more detail and specificity for the grades.

Following an introduction in chapter 1, chapter 2 elaborates six guiding principles for school mathematics instructional programs. An overview of the ten standards is provided in chapter 3. This is followed by grade band elaboration chapters for grades pre-K–2 (in chapter 4), grades 3–5 (in chapter 5), grades 6–8 (in chapter 6), and grades 9–12 (in chapter 7).

An eighth chapter is planned for inclusion in the final version of *Principles and Standards* that will focus on implementation and policy implications. This draft offers an outline of chapter 8, and readers are invited to submit comment and reaction.

The meaning of "standard" has not only a rich and long history outside of education, but a complex evolution in education in recent years. Despite some expectation that NCTM should establish performance standards in the sense of measurable benchmarks at each grade level. *Principles and Standards* does not provide that level of specificity of prescriptiveness. Rather, the standards presented here describe what students should know and be able to do in mathematics, and are statements about what is valued. The combination of principles and pre-K–12 standards here are intended as guidance to those who design, plan, implement, and evaluate practices, policies, and programs in mathematics education.

This draft of *Principles and Standards* includes a number of substantive and structural features that are different from the original *Standards* documents. Perhaps the most significant structural feature is that this product is presented in both print and electronic format. The production of *Principles and Standards* in electronic format, by the Electronic Format Group, will allow a richer array of examples to illuminate and extend the ideas provided in the text and will make possible the inclusion of links to resource and background material. The electronic version is available on CD-ROM and also on the World Wide Web at **<http://standards-e.nctm.org>**.

Chapter 1
Introduction

This is a time of extraordinary and accelerating change. The world is dramatically different from what it was in 1989, when the National Council of Teachers of Mathematics (NCTM) released the *Curriculum and Evaluation Standards for School Mathematics*. Consider as just one example the fact that student access to computers and the World Wide Web is now relatively common, whereas in 1989 availability of handheld scientific or graphing calculators was uncommon. School mathematics education bears increasing responsibilities in a data-rich era. Mathematics instructional programs should provide individuals access to mathematical ideas and should promote students' abilities to reason analytically. In a society saturated with quantitative information ranging from global climate change data to political polls and consumer reports, such skills will help students to understand, make informed decisions about, and affect their world. School mathematics education should contribute to the development of a public aware of the contributions of mathematics to society and capable of determining the social and economic consequences of their own decisions as well as those made by elected representatives on their behalf.

Building on the NCTM Standards Documents

The three NCTM *Standards* documents—the *Curriculum and Evaluation Standards for School Mathematics* (1989), the *Professional Standards for Teaching Mathematics* (1991), and the *Assessment Standards for School Mathematics* (1995)—offered teachers and policymakers new directions and visions for school mathematics. *Principles and Standards for School Mathematics* is intended to build on that solid foundation with a set of principles and standards that are focused, coherent, responsive, and grounded. Within this context, two key questions are addressed:

- What are the characteristics of mathematics instructional programs that will provide all students with high-quality mathematics education experiences across the grades?
- What mathematical content and processes should students know and be able to do as they progress through school?

Principles and Standards That Are Focused

In this draft of *Principles and Standards*, mathematics classrooms are viewed as places where thinking about and doing mathematics is the central focus. In such classrooms, students will be able to achieve the goals established in the *Curriculum and Evaluation Standards*: learning to value mathematics, becoming confident in their own ability, becoming mathematical problem solvers, learning to communicate mathematically, and learning to reason mathematically. In classrooms that focus on mathematical thinking, students will learn mathematics that is important in the pre-K–12 curriculum.

This document is built around ten standards. Within each standard, a small set of focus areas is identified, and these are elaborated in each of the grade-band chapters, as appropriate. With this set of standards, teachers and policymakers can build instructional programs and curricular frameworks that develop progressively over the grades and that focus on important mathematical areas for student learning.

Focus is also promoted in this document by the attention devoted to the idea of "moving on" in the curriculum, as concepts and procedures become part of students' repertoires. School mathematics instructional programs will not address every focus area within every standard during every year. Instead, students will achieve understanding of concepts and fluency with procedures at certain points in the pre-K–12 curriculum. Teachers should be able to assume these understandings and levels of fluency as they build their mathematical instruction. Students should make use of their understandings and procedural skill as they encounter other areas of mathematics or more sophisticated treatments of familiar areas.

A consequence of focus in this draft document is that the grade-band chapters cannot be comprehensive. Not every possible topic that might be included in instruction is listed. Rather, those topics that are of critical importance for instructional emphasis are offered as essential elements of mathematics education. A good instructional program will, of necessity, have to focus on the most fundamental ideas of mathematics. The standards advanced in this draft strive to provide the direction needed by teachers as they decide how, when, and how much to focus their instructional efforts.

Principles and Standards That Are Coherent

Chapter 3 of this draft provides an overview for each of the ten standards. Each overview describes focus areas within the standard as well as the development of student learning and understanding across grades pre-K–12. An elaboration of each standard for the specific grade bands is found in chapters 4–7. We hope that this structure provides coherence to the discussion of the standards.

Principles and Standards contains five mathematical content standards and five process standards. In some ways, this organization signals a more substantial emphasis on process standards than in the previous *Standards* documents. However, the process standards have been refined to express strong attention to mathematical learning. This document discusses the complicated interplay between mathematical processes and the learning of mathematical content and illustrates the integration of processes and content in understanding mathematics.

Finally, teachers and policymakers should find the treatment more coherent and practical in terms of offering guidance because ideas about curriculum, teaching, and assessment from the three original, separate documents are consolidated into a single volume. Principles about teaching and assessment are included; and, in the grade-band chapters, discussions of the process standards attempt to illustrate how mathematics learning looks in classrooms.

Principles and Standards That Are Responsive

A third intention of this draft of *Principles and Standards* is that it be responsive to the ten years of experience with the original *Standards* as well as to contemporary needs in mathematics education. With an emphasis on understanding mathematics that is fundamental, educators can be supported as they move beyond some of the superficial interpretations that have been made of ideas from the previous *Standards* documents. Currently, there are instructional programs that have emphasized solely some of the pedagogical intentions of standards—such as discourse, worthwhile mathematical tasks, or learning from problems—without sufficient attention to students' learning of mathematics content. Through a reformulation of process standards to focus more on mathematics learning and explicit detail about instructional emphases at the grade bands, the intention is to help teachers and policymakers expand their vision of effective mathematics teaching and learning to include a strong component about students' understanding of mathematics.

The emphasis on "mathematics for all" that was the hallmark of the *Curriculum and Evaluation Standards* has been interpreted by some as neglecting the needs of students who have deep interest in pursuing mathematical and scientific careers. This draft of *Principles and Standards* reaffirms NCTM's commitment to providing the highest-quality mathematics instructional programs for all students. Thus, this draft attempts to elaborate more fully the issues and classroom-specific possibilities for meeting the needs of students with diverse experiences and interests, including those who are especially inclined toward continued study of mathematics.

Given continuing rapid change in technology, both within and outside the school environment, this draft of *Principles and Standards* goes further than the original *Standards* documents in describing the role of technology. A principle about technology is now included. The discussion considers not only how technology might best support mathematics learning but also how the presence of technology implies shifts in mathematical content emphasis and the ways in which students' thinking might be qualitatively different. Also in response to changing technologies, this document is provided in electronic forma.

Principles and Standards That Are Grounded

Educational research, the judgment of professionals, and the accumulated experiences of professional teaching practice are three domains that play a role in the formulation of educational standards. This draft is well-grounded in the sense that the ideas included in *Principles and Standards* draw on important elements from these domains.

Educational research serves as a basis for proposals and claims made throughout this document. In addition, research methodologies and perspectives have contributed to the ways in which focus areas of some of the standards are framed and described. Research also is helpful in determining what is possible for students to learn about certain content areas at certain levels and under certain pedagogical conditions.

In contrast, the actual content choices offered in any set of standards reflect values, together with practical and historical considerations. As one educator puts it, "Research cannot select the standards" (Hiebert 1998, p. 4). The selection of content and process emphases in *Principles and Standards* is based on the views provided by professionals in mathematics and mathematics education about societal needs for mathematical literacy, past practice, and expectations of the general public for school mathematics

A third way in which this draft is grounded is in its consideration of the accumulated wisdom of professional teaching practice. The vignettes and classroom episodes included throughout the document are composites drawn from the experiences and observations of the classroom teachers, teacher educators, educational researchers and mathematicians in the Writing Group, and from input that we received. Although the draft of *Principles and Standards* deals primarily with issues of mathematics teaching and learning in classrooms, it also recognizes the recent shifts in thinking within the education field toward systemic change. Mathematics classrooms are embedded in larger contexts, including schools, districts, and state, provincial, and national initiatives. So, the principles described in this text are introduced as a way of highlighting the interplay between classrooms and the larger system in seeking improvement of mathematics education.

Principles and Standards for School Mathematics

This draft of *Principles and Standards for School Mathematics* is organized into three parts:

- Guiding Principles for School Mathematics Instructional Programs (Chapter 2)
- Overview of Standards for Grades Pre-K–12 Mathematics Education (Chapter 3)
- Standards for Grades Pre-K–2, 3–5, 6–8, and 9–12 (Chapters 4–7)

The organization is illustrated in figure 1.1.

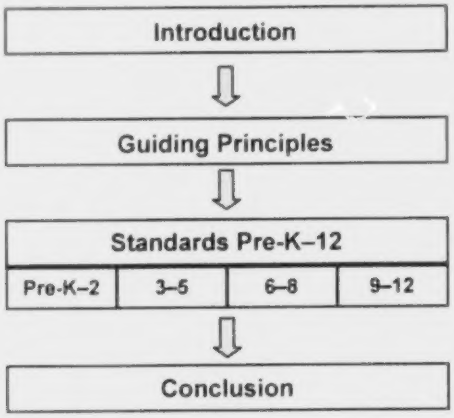

Figure 1.1. Structure of *Principles and Standards* document.

In responding to NCTM's charge, the Writing Group has made a commitment to the learning of deep and powerful mathematical ideas by all students, as expressed in the ten standards that form the core of this draft document. Surrounding and supporting these ten standards is a set of principles for school mathematics instructional programs that address equity, mathematics curriculum, teaching, student learning, assessment, and technology, as represented in figure 1.2. The principles are offered as basic tenets on which to establish quality programs and guide decisions about mathematics instruction at the classroom, state or provincial, or district levels and beyond.

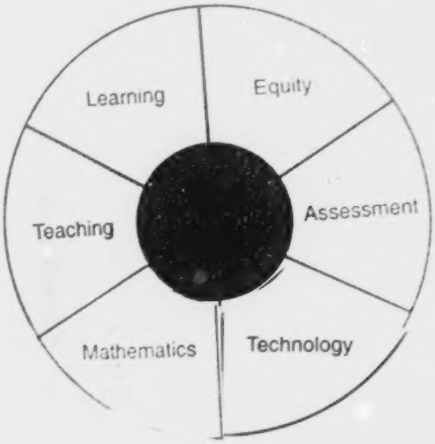

Figure 1.2. Principles and Standards for School Mathematics

The standards for grades pre-K–12 mathematics education are designed to apply across all grades. There are five content standards which address the areas of number, measurement, algebra, geometry, and data. Each content standard describes goals for students' understanding of concepts and procedures organized by a few focus areas. There also are five process standards, which address problem solving, reasoning, connections, communication, and representation. The process standards are described in terms of students' mathematical learning outcomes. The "Standards Overviews" included in chapter 3 provide a sense of the developmental trajectory of the mathematical content or process area. They describe how the area will grow more sophisticated in depth and breadth over the grades and highlight at which points certain levels of mastery—or "closure"—are appropriate. The overviews also address crucial transition points in students' learning along the way.

The mathematical content areas identified within the original *Standards* documents are preserved in this update. Note, however, that the *Curriculum and Evaluation Standards* highlighted a number of topics under the heading of discrete mathematics—topics that have become important for school mathematics because of the increased availability and use of technology. At the time the original *Standards* were written, this highlighting was important, but it has since led to some interpretation of discrete mathematics as a completely separate strand of mathematics, to be taught in addition to traditional topics. In this draft, major topics from discrete mathematics have been integrated into the other strands. For example, the development, use, and analysis of algorithms is central to the study of number systems, algebra, geometry, and data and recursion is emphasized in algebra. Note also that a standard on "representation" is added because of the need to highlight the importance of such processes as organizing, recording, and communicating mathematical ideas and using representation in modeling.

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Four grade-band chapters follow the standards overviews, organized into the grade-bands pre-K–2, 3–5, 6–8, and 9–12. In each grade-band chapter, the ten standards are elaborated in terms of key focus areas, with varying emphasis and depth as appropriate. The grade-band chapters introduce more specificity to the focus areas of the standard, with attention to when certain dimensions are completed or when new components enter the curriculum for students. Although the standards presented in this draft document are designed to extend across the entire pre-K–12 spectrum, certain components of each will be more prominent and important in particular grade-bands. The grade-band sections also introduce examples and images of classroom practices and context.

#1

Reader Reaction

The final draft of *Principles and Standards for School Mathematics* needs to be considerably shorter than this draft. Can you suggest how we might eliminate redundancy and express our message more concisely?

Please indicate places in the draft where you feel additional references would be helpful, and suggest specific references if you have them.

Chapter 2
Guiding Principles for School Mathematics Instructional Programs

Mathematics learning is a matter of central importance to our society. Mathematical experiences in schools enable students to become mathematically literate citizens. How much mathematics students learn, and how well they learn it, depends to a great extent on dimensions of the quantity or quality of the mathematics instructional programs they encounter. The opportunities made available to students place upper bounds on how much mathematics and what kinds of mathematics can be learned. The instructional materials used by teachers have great impact on how much students learn. Whether or not teachers, parents, and administrators convey high expectations of their students also influences how much students learn. Even the requirements set by states, provinces, or districts about high school graduation speak to expectations. When students are challenged in appropriate ways, they respond.

Decisions made by teachers, school administrators, and other education policy professionals about the content and character of mathematics instruction encountered in school have important consequences for students. Those students who have many opportunities to study well-taught, important mathematics are more likely to gain mathematical proficiency—and the associated educational and employment advantages—than students who have fewer such opportunities. Much can be done to increase the likelihood that learning opportunities become learning outcomes. Students' mathematics learning can be supported when instruction regularly emphasizes meaningful engagement with cognitively demanding tasks (Stein and Lane 1996); when teachers regularly utilize the computational, graphic, or symbolic capabilities of technological tools to develop mathematical ideas (Carpenter 1998; Heid 1988; Hiebert and Wearne 1996) and when teachers are supported by professional development programs that refresh and enhance their knowledge of mathematics content, pedagogy, and student learning (Campbell 1995; Cohen and Hill 1997; Fennema, et al. 1996).

Mathematics Instructional Program Principles

The following principles describe basic tenets about high-quality mathematics instructional programs. They offer education professionals a guide for making decisions that influence students' learning opportunities. The principles apply at many levels of the educational system and thus encourage and support systemic change. The principles also express the perspectives and assumptions that underlie the ten standards discussed in chapter 3.

Equity Principle: Mathematics instructional programs should promote the learning of mathematics by *all* students.

Mathematics Curriculum Principle: Mathematics instructional programs should emphasize important and meaningful mathematics through curricula that are coherent and comprehensive.

Teaching Principle: Mathematics instructional programs depend on competent and caring teachers who teach all students to understand and use mathematics.

Learning Principle: Mathematics instructional programs should enable all students to understand and use mathematics.

Assessment Principle: Mathematics instructional programs should include assessment to monitor, enhance, and evaluate the mathematics learning of all students and to inform teaching.

Technology Principle: Mathematics instructional programs should use technology to help all students understand mathematics and should prepare them to use mathematics in an increasingly technological world.

Together, the principles and standards presented in this draft document can be useful for identifying directions to follow and suggesting course corrections for educators as they progress on the path of steady, incremental improvement of mathematics education.

#2

Reader Reaction

As you read the next sections, please consider the following:

- **What is your reaction to these principles as a set? Would you have deleted some? Chosen others?**
- **Does each principle describe clearly the basic tenets and assumptions for school mathematics instructional programs within that area? Where do we need more detail or clarity?**
- **Do you have questions or disagreements with the ideas discussed here?**

The Equity Principle

Mathematics instructional programs should promote the learning of mathematics by *all* students.

A commitment to equity was one of the hallmarks of the original NCTM *Standards*, and its importance is reaffirmed here. One of the most notable aspects of NCTM's 1989 *Curriculum and Evaluation Standards for School Mathematics* was the insistence that mathematics instructional programs address the learning of mathematics by *all* students.

An emphasis on "mathematics for all" is important because of the role that school mathematics has historically played in educational inequity. A student's mathematical proficiency is often used as a basis for decisions regarding further schooling and job opportunities. Furthermore, mathematics has been one of the school subjects frequently associated with "tracking," a practice in which students are sorted into different instructional sequences that often results in inequitable educational opportunities and outcomes for students.

An emphasis on "mathematics for all" is also important because it challenges a pervasive belief among some members of society that a great number of students are not capable of acquiring mathematical proficiency. This belief is quite different from the equally pervasive view that all students can learn to read and write in English. Verbal literacy has been expected of all students, and school programs that do not produce literate students are regarded as failures. It is striking that a contrary set of beliefs has traditionally been held regarding mathematical literacy and proficiency. The education system has been expected to succeed only with the "mathematically able."

To underscore the need for a high-quality mathematics instructional program for all students, the *Professional Teaching Standards for School Mathematics* explicitly stated as one of its undergirding assumptions that "all students can learn to think mathematically" (NCTM 1991, p.21). By "all students," the authors meant—

- students who have been denied access in any way to educational opportunities as well as those who have not;
- students who are African American, Hispanic, American Indian, and other minorities as well as those who are considered to be part of the majority;
- students who are female as well as those who are male;
- students who have not been successful as well as those who have been successful in school and in mathematics. (ibid., pp. 21-22)

The importance of equity was also emphasized in NCTM's 1995 *Assessment Standards for School Mathematics*. One of the six mathematics assessment standards was the Equity Standard, "Assessment should promote equity" (p. 15).

Striving for Equity and Excellence

In a democratic society, a commitment to equity should be of central importance in mathematics education. Traditionally, school mathematics has been viewed as a sorting machine, in which many students are considered unlikely to study higher mathematics and in which a few students are identified as capable of succeeding in the discipline of mathematics or in mathematically related fields of study. Arguably, for a society in which the mathematical needs of most citizens are quite low, this form of mathematics education might be adequate, even if not equitable. In recent decades, however, more and more citizens—most of whom would not be identified as "mathematically talented"—need to comprehend and respond competently to quantitative information. Consider our current culture, in which banks have become increasingly "self-service" with ATM machines and on-line banking. Widely available consumer reports offer information on product safety and effectiveness. Employment opportunities in fields such as health care, auto manufacturing, news reporting, and fashion design require a more sophisticated knowledge of mathematics. As a consequence, a "sort and discard" form of school mathematics education is inadequate.

An alternative conception of school mathematics instructional programs assumes the primary purpose of ensuring that all citizens are quantitatively literate. That is, all students must become fluent and flexible in dealing with quantitative information. This view goes well beyond arithmetic computation and algebraic equation solving, which are often thought to be the hallmarks of mathematics education for all students. Moreover, a commitment to equity should support another central goal of mathematics education—namely, excellence. Achieving these two goals is a complex and interrelated task. Equity will be achieved when excellent mathematics instructional programs exist for every student in every school.

School mathematics instructional programs need to nurture the development of mathematical proficiency and quantitative literacy in students who are exceptionally talented and in those who will be productive, quantitatively competent citizens. If school mathematics instructional programs serve their purpose well, there will be no shortage of students well prepared to pursue further education and careers in mathematics, science, and engineering, and all citizens will have adequate quantitative skill.

Believing and Acting

Adopting the belief that all students can learn mathematics is critical. But it will take more than good intentions to enact equitable mathematics instructional programs. As stated above, one impediment is the widely held belief that only some students can succeed in mathematics. Low expectations are especially problematic because they are not randomly distributed across the entire population. Females, students who live in poverty, students who are not native speakers of English, and many non-white students have traditionally been far more likely than their demographic counterparts to be the victims of low expectations.

Lower expectations can manifest themselves in both obvious and subtle ways in the elementary school—in patterns of classroom instruction and interaction, in the grouping of students for instruction, and in differential assignments. In the secondary school, low expectations are communicated in these ways and also in the enrollment of students in low-level courses with minimal objectives and payoff and in advising these students away from more demanding elective courses or career objectives. It is essential that expectations be raised for all students, especially for students in demographic groups that have in the past been underserved by school mathematics education.

Inequity is pervasive, but it can be corrected in mathematics instructional programs. Consider the case of gender equity, which has been a concern in mathematics education for at least several decades. In the 1960s and 1970s, numerous studies and surveys indicated a substantial gap between males and females with respect to mathematics participation and performance. In the past two decades, great progress has been made in addressing this issue (Fennema and Leder 1990; Leder 1992). Although gender inequity still exists in some schools and mathematics classrooms, it is clear that the "gender gap" has been substantially reduced. For example, males and females now enroll in college preparatory mathematics courses and in elective, advanced courses at the secondary school level in about equal numbers, and their mathematics achievement on many measures is quite similar. Similar results have also been noted in the elementary and middle grades on most measures of mathematics achievement.

Improvements with respect to gender equity in school mathematics have resulted from a combination of higher expectations on the part of many persons—teachers, teacher educators, developers of curriculum and supplemental instructional materials, organizers of special programs, school administrators, researchers, parents, and community members—and the consistent and sustained efforts over a long period of time to address the sources and symptoms of inequity. Continued efforts in this regard are still needed, but substantial progress has been made. An increase in expectations, accompanied by a similarly robust and concentrated effort, is now needed to address inequities associated with factors such as ethnicity and poverty.

Schools in middle- and high-income communities are typically able to attract better qualified mathematics teachers than schools in low-income communities, and teachers in more affluent communities are more likely to have access to a variety of instructional materials and good professional development opportunities. Teachers and students in middle- and high-income communities often have ready access to role models who can demonstrate the value of schooling in general and mathematics education in particular. Students in these communities also often have easy access at home to academic support materials, such as calculators, computers, and supplemental books. Some research suggests that the cultural and family norms of families in middle- and high-income communities tend to be more closely aligned with school expectations for students and parents than is typically the case in low-income communities (Oakes 1990; Secada 1992; Singham 1998). All students, including those from poor communities, bring strengths from outside of school that should be recognized and rewarded. For example, many students who reside in poor communities and whose first language is not English find their understandings assessed in school only in English. This practice ensures that school assessments often underestimate the mathematical proficiency of non native speakers of English. As was true for gender, many of the sources of inequity reside outside the control of schools. But there is much that can be done by teachers and other educators.

Research-based examples exist to demonstrate the validity of the view that all children, including those who have been traditionally underserved, can learn mathematics when they have access to quality mathematics instructional programs (Campbell 1995; Griffin, Case, and Siegler 1994; Knapp et al. 1995; Silver and Stein 1996; Silver 1998). Actions are needed to make these examples the norm rather than the exception in school mathematics education. Data that are disaggregated by ethnicity, language, gender, or social class can be very useful in examining the effects of district and school policies and teachers' classroom practices and in developing plans to address the problem. At the secondary school level, for example, such analyses could help to detect patterns of differential course offerings between and among schools; differential enrollment in elective courses within a school; or differential achievement within a classroom, school, district, state, or province. States, provinces, school districts, schools, and classroom teachers all should examine potential inequities associated with mathematics instructional programs and act on them.

Holding High Expectations and Providing Strong Support

Higher expectations alone are not sufficient to accomplish the goal of an equitable school mathematics education for all students. Along with higher expectations, many students will need greater support in their mathematics instructional program. All students need access to a coherent, challenging mathematics curriculum that is taught each year by a competent and well-supported mathematics teacher. Moreover, students' learning and achievement need to be assessed and reported in ways that attend to a reasonable range of important dimensions of mathematics and mathematical performance. The suggestions that are made in the following principles about mathematics curriculum, teaching, learning, assessment, and technology are applicable to all students.

In order to achieve educational equity, it is critical that students who need the most assistance in learning mathematics have access to an excellent mathematics instructional program that provides solid support for their learning and is responsive to their prior knowledge, intellectual strengths, and personal interests. Students who have been underserved by traditional mathematics instructional programs need this and more. Instructional programs that address both the challenges and strengths that students bring with them to the classroom are more likely to be successful. In addition to excellent teaching, some students also may need additional assistance with their mathematics work, such as peer or cross-age tutoring and after-school programs. Others may need social supports, such as role models who demonstrate the possibilities for success. Students who arrive at school speaking a language other than English may need special attention in order to ensure that they can participate fully in classrooms in which mathematical discourse is central. All students should be able to use their own language to develop mathematical ideas as they simultaneously develop proficiency in English.

Technology can be of assistance in achieving equity. For example, technology provides access to hardware and software tools and environments that afford opportunities for exploring complex problems and new mathematical ideas; give structured tutorials to students needing additional instruction and practice on skills; or link students in rural communities to instructional opportunities or intellectual resources not readily available in their locales. The most interesting new mathematical ideas that are suggested by developments in technology must be part of school mathematics programs for all students. Moreover, it is critical that all students have opportunities to use technology in appropriate ways in the mathematics classroom. Access to technology must not become yet another dimension of educational inequity associated with mathematics .

The Mathematics Curriculum Principle

Mathematics instructional programs should emphasize important and meaningful mathematics through curricula that are coherent and comprehensive.

The opportunities provided to students to learn mathematics determine the limits of what students may actually learn. Mathematics curriculum as enacted at multiple levels—teacher lesson plans, a year-long instructional program, school or district curriculum guidelines, and state or province frameworks or standards—provides the basis from which teachers make decisions about what content to address, what emphases to choose, what pedagogical strategies to employ, and what assessment practices will be appropriate. Mathematical curricula have profound influence on what students will know and be able to do in mathematics.

Studies such as the Third International Mathematics and Science Study (TIMSS) (Schmidt et al. 1996) highlight the fact that curricula in the United States tend to include many more topics than the curricula in other countries for a given year. Moreover, even though fewer topics are addressed each year in other countries, the study revealed that by the end of secondary school at least as many topics are covered in other countries and in greater depth than in the United States. This finding demonstrates that decisions about which topics are included, when they are introduced, and in what depth have a strong impact on the success of school mathematics instructional programs.

Those who make decisions and choices about the mathematics curriculum at all levels must grapple with a number of complexities and challenges. In the language of the original *Standards* documents, the mathematics included in school programs should be sound, significant, and important. Areas of mathematical content and process selected for inclusion need to be arranged into coherent, comprehensive curricula that foster mathematical development over time. Teachers and other decision makers need increased understanding and guidance to examine and judge curricula, especially such aspects as the importance of the mathematics, the coherence and comprehensiveness of the design, and the potential for enabling understanding of mathematical ideas by all students.

As in the original *Standards*, the term "curriculum" as used here incorporates several dimensions, including the mathematics that students need to know, how students are to achieve mathematical goals, what teachers need to do to help students develop their knowledge, and the context in which teaching and learning occur (NCTM 1989). The Mathematics

Curriculum Principle focuses only on the *mathematical* elements of curriculum. Matters of teaching and assessment are taken up in the principles discussed later in this chapter.

Choosing Mathematics That Is Most Important

In considering whether the mathematical tasks for a lesson, topics in a textbook, or goals in a framework are important, a number of criteria are relevant. The criteria involve several dimensions: the place of the topic within mathematics itself; the role of the topic outside of mathematics; and the relevance of the topic for the learner.

Mathematical content and processes are important in school mathematics instructional programs if they are fundamental to deeper mathematical study. Concepts such as place value, function, scaling and similarity, structure in the number system, and rate of change qualify as fundamental in that they are basic and essential to understanding a variety of areas of mathematics. Procedures such as solving equations or graphing functions are fundamental for the same reason. Mathematical processes such as making sound arguments or knowing when and how to use technology are also fundamental. Frequently used basic facts and definitions are fundamental in that they support problem solving and mathematical communication. These typify mathematical areas that are springboards for deeper and broader mathematical content.

Mathematical content and processes are important for school mathematics instructional programs if they are useful—either for developing other mathematical ideas, linking areas of mathematics, or forming and solving problems within or outside of mathematics. Citizens in today's society require a higher level of quantitative skills than ever before, including abilities to assess claims, detect fallacies, evaluate risks, and weigh evidence (Price 1997). Mathematical topics and emphases such as recursion, iteration, and the comparison of algorithms become possible and relevant for the school curriculum because of their utility in a technological world. Mathematical thinking and reasoning skills that can be applied in a range of situations outside of school are useful, as are the basic facts and computational skills needed in doing the daily mental calculations and estimations that all citizens encounter.

Mathematical content and processes are important if they build productively on students' prior knowledge and experience, engage their interest, and make sense. A challenging pedagogical task for teachers is finding ways to connect new mathematical ideas to the understandings and experiences that students bring to the classroom. Choosing problem contexts that might be typical of what students find in their worlds, however, is but one way of engaging students effectively. Engagement can also grow out of contexts that are purely mathematical, either proposed by the teacher or generated through student questions and curiosity.

Judgments about curricular quality are needed not only when schools or districts are selecting materials, but also when classroom teachers are planning for a year, a unit, or a lesson. Central to this planning process are choices about mathematics content. NCTM's *Professional Teaching Standards* (1991) provides guidance to teachers for the selection, adaption, and generation of mathematical tasks by suggesting that they consider the content, the students, and the ways in which students learn.

A mathematics instructional program is *coherent* when the mathematics in it is well chosen and effectively organized and integrated. Coherence will be visible in the transitions made in moving to new topics, in the ways in which earlier topics are used for later topics, and in the consistency of style and approach.

Many adults who recall their school mathematics experiences as less than satisfying see mathematics as a set of disconnected facts and procedures. These adults often observe that once the burden of recalling these separate facts and procedures in school became too great, their performance in mathematics declined, and they never recovered. Conversely, mathematicians see their subject as both highly interconnected and cumulative. Yet, their image of mathematics as a subject that builds and grows through the interconnection of ideas is not always readily apparent in school textbooks or within lessons in the classroom. It is important that students see the interconnections of ideas in the subject of mathematics through their experience in school mathematics.

Coherence is important at every level of the curriculum, including the individual classroom lesson. An important characteristic studied by researchers as they analyzed lessons in the TIMSS videotape study of mathematics classrooms (Stigler and Hiebert 1997) had to do with the internal coherence of the mathematics contained in the lesson. The researchers found that typical Japanese lessons were designed around one central idea, which was carefully developed and extended. In contrast, typical American lessons included several ideas or topics that were not closely related and not well developed. Students will better acquire and utilize mathematical knowledge from coherent curriculum.

Mathematics curriculum should include content and processes that make up a *comprehensive* set of instructional goals and activities. The set of mathematics standards proposed in this document is constructed to be comprehensive, in the sense that these standards should constitute the core curriculum for all students.

A comprehensive curriculum should achieve an appropriate balance of conceptual knowledge and procedural competence in important areas. Research provides some guidance on how to achieve this balance. For example, problem solving can be used effectively as a context in which students will learn new concepts and skills (Hiebert 1998), and conceptual understanding can precede skill development in a technological environment (Heid 1988; Palmiter 1991). Excessive practice at procedures before understanding can lead to difficulty in making sense of the procedures later (Hiebert 1998).

Another aspect of a comprehensive curriculum is how well that curriculum affords students opportunities to learn about the nature and practice of mathematics. Students need to see that mathematics is a human enterprise that, although often abstract, has tremendous power in explaining and predicting real-world phenomena. They also need to see that mathematics is based in rigorous systems of definitions, axioms, and theorems. Students also come to understand through experience that reasoning often requires considerable time and experimentation with arguments that fail before finding one that succeeds.

Curriculum should be designed to systematically provide students with mathematical experiences that become progressively deeper and broader. When mathematical content and processes recur through the grades, they should be experienced at deeper levels. For instance, students in grade 2 may be introduced to bar graphs by counting M&M's by color and representing the results. In a later grade, if the same activity of counting by color is used again, teachers should have a clear idea about what deeper understanding they are hoping students will achieve, rather than repeat the construction of bar graphs.

The problems posed and the concepts examined within any mathematics area should grow more sophisticated each year of the curriculum until a suitable level of understanding or proficiency is reached. At that point, although formal instruction in the specific content area will end, students should continue to use the understandings and skills they have acquired, thereby maintaining and strengthening both. For example, students should understand and be fluent with whole-number computation by the end of grade 5. This proficiency forms the foundation on which students will work with other sets of numbers and make calculations in algebra, as arithmetic calculation recedes from its prominent position in the program.

The Teaching Principle

Mathematics instructional programs depend on competent and caring teachers who teach all students to understand and use mathematics.

More than any other single factor, teachers influence what mathematics students learn and how well they learn it. Students' mathematical knowledge, their abilities to reason and solve problems, and their self-confidence and dispositions toward mathematics all are shaped by teachers' mathematical and pedagogical decisions. The Teaching Principle emphasizes the many responsibilities of the teacher in creating the mathematics classroom as a place for thinking and learning. Such teaching can be accomplished only by teachers who have had strong preparation and who will have ongoing access to effective professional development.

The *Professional Teaching Standards* (NCTM 1991) describes the major arenas of teachers' work: analysis of and reflection on teaching and learning, worthwhile mathematical tasks, the learning environment, and classroom discourse. These aspects of teaching are critical to the development of mathematical knowledge for all students and are amplified here through discussion of—

- how analysis and reflection about student learning, mathematical goals, and the classroom environment should be the basis for teachers' decision-making and planning;
- what makes mathematical tasks worthwhile, and how they fit into a coherent curriculum;
- ways in which learning environments foster mathematical dispositions and communities;
- how discourse supports the learning of important mathematics.

Analysis and Reflection

Thoughtful analysis and reflection are essential to successful teaching. From decisions about long term goals to planning a unit or a lesson or reacting to student questions, teachers constantly make decisions about mathematical content and pedagogy. In making decisions, teachers rely on their knowledge of mathematics, their understanding of students as learners, their experience in knowing what kinds of questions and misunderstandings students are likely to have, and on the mathematical learning goals they have identified for their students. Planning, teaching, and assessing should be linked through teachers' ongoing analysis about what is working, what they know about the students, and what constitutes learning of important mathematics. In particular, teachers apply all of this knowledge in the context of practice in classrooms to decide what to do in the short and long term (Ball 1993).

Worthwhile Mathematical Tasks as Part of a Coherent Curriculum

In order to be sure that mathematical tasks fit together into a lesson, unit, or yearlong program that has coherence and that builds mathematical ideas, teachers need to consider the relationship of a task to the broader mathematical goals. There are many tasks that students find interesting, but not every interesting or motivating task will fit well into the goal of promoting the learning of important mathematics in the areas identified in these standards. To make judgments about tasks, teachers need to know mathematics well beyond the mathematics they teach.

Determining what students understand, recognizing their insights and their misconceptions, and their students' knowledge are part of teaching practice. Lessons should be designed with thoughtful anticipation of students' questions and responses to the tasks. Since students build new understandings as they make connections with prior experiences, they often refine and clarify their knowledge about old ideas as they encounter new ones.

Worthwhile tasks are ones that students can access on different levels and that challenge different students in different ways. Good tasks fuel students' curiosity and encourage them to talk about mathematics. Tasks must be selected within the context of nurturing mathematical learning. Deciding what aspects of a task to highlight and what questions to ask that will challenge rather than confuse students' thinking requires careful preparation of lessons and attention to discourse within the classroom.

In selecting mathematical tasks, teachers also need to consider which curriculum materials will best support the designated goals, what technologies will enhance students' understandings, and how to organize instruction to focus students on the mathematical ideas. Effective teachers use strategies and approaches that range from extended student explorations in small groups to direct telling. They shift among strategies as the students' needs shift, and they mix strategies to allow for all students to have effective experiences. Decisions about which instructional strategy to use must be based on what mathematical ideas are being taught, the nature of the tasks, and the status of students' understanding.

Learning Environments to Foster Mathematical Dispositions

Safe classroom environments provide support for students to become mathematical thinkers. The classroom environment communicates subtle messages about what counts in learning and doing mathematics. Teachers bear the primary responsibility for establishing a classroom atmosphere that sends a clear message that they believe that each student can learn important mathematics and that they are determined to help students accomplish this goal.

Teachers' dispositions help shape students' dispositions by the overt and subtle messages they convey. Teachers who display enthusiasm for challenging mathematics and emphasize sense-making and mathematical competence are likely to foster similar attitudes and dispositions in their students. When students see teachers exploring alternative ways of approaching tasks and posing provocative questions to the class, they are likely to experiment with alternative strategies and ask questions of themselves and their classmates. When students see teachers summarize or reflect on the mathematics that has been learned or point out what is common across several student solutions, they too will begin to see important elements and connections.

In applying familiar concepts in different situations or in learning new mathematics, students may not at first know how to proceed. If teachers choose tasks that will challenge students' thinking, then they must also make it comfortable for students to start down the path of solving problems knowing that their first steps may not be correct. When first conjectures prove not to hold, students need to know that teachers expect them to invest time and energy in continuing to work on the problem and that teachers will support them in their efforts. Determining what support to give students is among the most important decisions teachers make, for there is a fine balance in promoting autonomy, allowing students to struggle when they may not be headed in productive directions, and giving specific guidance.

Teachers must consider goals for the class as a whole as well as for individual students. All classrooms are likely to have students with different background experiences, interests, and dispositions that must be accommodated. Helping all students engage productively in a mathematical task often requires customization of the task for individuals, in terms of the task itself and the levels of understanding and proficiency involved. By using a variety of assessment strategies—including questioning, conversations, and students' written work—teachers can gain insight into students' thinking and reasoning and can capitalize on what students already know in building a productive learning environment for all.

Discourse to Support Mathematics Learning

In the past decade, as teachers have made efforts to attend to discourse in their practice, they have discovered the mathematical and pedagogical challenges in this type of teaching. Students can be actively engaged in lively classroom discourse that is superficial and not rich in mathematics if teachers are not able to make careful decisions about how to design and orchestrate student activity. Just as incorporating manipulatives, calculators, and computers into lessons may assist students in expanding their understandings, but alone may not lead to mathematics learning, having students talking and writing about mathematics does not automatically provide them with increased mathematical understandings. Recognizing how to direct discussion and how to balance telling, leading, asking, and making mathematical summaries are important skills for the teacher who is building a mathematical community in the classroom. Knowing the capacity of the entire class to move forward from a single student's idea and judging the tradeoffs in allowing an unanticipated digression are part of orchestrating good mathematical discourse.

In conclusion, teaching mathematics well and creating classrooms where mathematical thinking is central is difficult and rewarding work. Curricular frameworks and guides, instructional materials, and lesson plans are only the first elements needed to help students learn important mathematics well. Teachers must balance purposeful, planned classroom teaching with the ongoing decision-making that can lead the teacher and the class into unanticipated territory from an effective mathematical and pedagogical knowledge base. Teachers can provide classrooms that promote thinking, but it takes much more than worthwhile mathematical tasks and a commitment to discourse. It takes deep insight about mathematics, about teaching, and about learners, coupled with a sound and robust mathematics curriculum and thoughtful reflection and planning.

The Learning Principle

Mathematics instructional programs should enable students to understand and use mathematics.

This draft of *Principles and Standards for School Mathematics* builds on the vision of learning as sense-making that was presented in earlier *Standards* documents. Here, as in the original documents, mathematics learning is inextricably tied to understanding and use. The notion of mathematics as something to be deeply understood, so that it can be used effectively, has not always been a valued outcome of school mathematics. Many adults are quick to admit that they are unable to remember much of the mathematics they learned in school. In their schooling, mathematics was presented primarily as a collection of facts and skills to be memorized. The fact that a student was able to provide correct answers shortly after studying a topic was generally taken as evidence of understanding. Students' ability to provide correct answers is not always an indicator of a high level of conceptual understanding . Understanding is not something that one either has or does not have. Rather, understanding varies in depth and an individual's understanding of any idea is constantly changing. Understanding of mathematical ideas should grow and deepen across the grades.

Building on Prior Experience and Making Connections

Mathematics is developed through human activities—problem solving, reasoning, representing, communicating, and making connections. From a very young age, children are interested in mathematical ideas and they gradually develop a rather complex set of informal ideas about numbers, patterns, shapes, data and size through their experiences in everyday life. Thus, children learn many mathematical ideas quite naturally even before they enter school. And, as learners interact with ideas and solutions to problems, they should develop more complex understandings. New learning is best built on prior learning, which can include ideas developed through school-based lessons, as well as informal intuitions developed through experience. Students of all ages bring to mathematics class a considerable knowledge base upon which to build.

The large literature on thinking and learning indicates that "networks" of ideas and the connections among them reflect the structure of conceptual organization; old knowledge shapes the way that new information is understood (see, e.g., Schoenfeld, Smith and Arcavi 1993; Siegler and Klahr 1982). Mathematics makes more sense, and is easier to remember and to apply, when students can find connections with existing knowledge. An image of learning where each learner adds knowledge onto nodes of his or her own individualized web of understanding at different points, at different rates, and in different ways, can be helpful. And, because different students bring different experiences and prior knowledge to any mathematics classroom, teachers must become expert at assessing student understanding and building on it in purposeful ways.

Another view of learning mathematics is to see it as a process by which people gradually become members of a mathematical community (Lave 1991; Sfard 1998). The classroom is a community of mathematical inquiry and the students are participants in that community, striving toward mapping and understanding mathematical ideas, norms, and rules. This view of learning presumes that a part of the process involves interactions with others as a means of testing out mathematical ideas, learning to evaluate the thinking of others, and develop mathematical reasoning skills.

Perhaps a blend of these two perspectives is most useful, depending on the focus. When assessing the knowledge of an individual student, teachers may focus on the "network" of knowledge held by that particular student, building on it and looking for missing links and for places from which new knowledge might be built. When assessing the over-all progress of a class, the teacher may need to closely consider the effectiveness of that learning community in moving forward with important ideas. These two perspectives are clearly interactive, as the learning community is made up of individuals, and individual knowledge is substantially shaped by the interactions that take place within the mathematics classroom.

Learning Concepts and Procedures

Some lines of research suggest that people do not generally learn concepts by building up pieces of knowledge (e.g. Resnick 1987). Instead, they are more likely to plunge headlong into a problem situation, bringing whatever facts, procedures, and understandings they have at hand. Thus, expecting students to master a set of so-called prerequisites in a prescribed order before they attempt to tackle challenging problems is not necessarily productive. Although students taught in this manner may learn facts or procedures well, they often are not sure when or how to apply them, and they also may forget them quickly. More importantly, these students have little chance to experience satisfaction that comes from grappling with complex and interesting tasks.

A focus on learning mathematics with understanding should not be taken to mean that there is no place in school mathematics for the learning of routine procedures. Elementary students who thoroughly understand the meaning of addition but cannot fluently retrieve the basic addition facts will be at a serious disadvantage as they grapple with ideas of multiplication and division. Secondary students must possess more than a basic conceptual understanding of equations and how they are solved; without facility to symbolically manipulate reasonable equations, little progress can be made in exploring more complex situations. Fluency with basic procedures at various levels is a key element for the continued building of mathematical knowledge.

There can be no doubt that both conceptual understanding and procedural proficiency are important. It is not the primacy of either that we should be considering. Instead, it is the connections between them that are important. For example, teachers may initially encourage their students to develop their own strategies for solving problems in ways that make sense to them. But, over a period of time, they need to pose problems that promote the development of increasingly efficient strategies, so that the students eventually are able to fluently solve such problems. On the other hand, teachers need to ensure that their students understand the sense behind the manipulations they learn. The absence of such understanding quickly leads to such all-too-familiar student claims as $100 \times 3.45 = 3.4500$ because to "multiply by 100 you just add on two zeroes" or $(x + y)/x = y$ because "you can just cancel the x's". It is essential that students' conceptual understanding and procedural proficiency are integrally connected.

The Importance of Disposition and Engagement

Learning mathematics also includes developing a disposition to see mathematics as a powerful way for looking at situations. Students' mathematical dispositions are manifested in the way they approach mathematics learning. Students need to be confident in their ability to tackle difficult problems, eager to figure things out on their own, flexible in exploring mathematical ideas and trying alternative solution paths, and willing to persevere when tasks are challenging. They should be inclined to monitor and reflect on their own thinking and to learn from their mistakes. A major goal of school mathematics instructional programs should be to create autonomous learners. Classrooms must promote the abilities of students to initiate and sustain inquiry by becoming mathematical communities where students exchange ideas and deepen conceptual understanding. As described in *Everybody Counts*, "To understand what they learn, [students] must enact for themselves verbs that permeate the mathematics curriculum: examine, represent, transform, solve, apply, prove, communicate" (NRC 1989, 58-59).

Active Engagement with Mathematics

Active engagement with mathematics is best fostered through problems that are motivating and challenging for students. Students' realization that their present knowledge is insufficient may provide further incentive, in addition to the motivation inherent in problems with interesting mathematical or real-world contexts. The use of manipulatives, technology, or other tools is certainly one way for students to be actively involved in learning mathematics, but so are a variety of other classroom activities, such as personal involvement in lively discussion, active listening to a lecture, or individualized or collaborative immersion in problem-solving activity

Teachers and students together share responsibility for mathematics learning. Each student is responsible for making sense of mathematics. The development of deeper understanding and meaning succeeds through a process of struggling with new concepts and incorporating that information into existing knowledge. With that expectation in mind, students will view the challenge of mathematics as a normal part of learning, rather than as a signal to give up and consider oneself a failure. Learning mathematics may not always be fun, but it can be engaging and rewarding. When students successfully solve a difficult problem or finally understand a complex idea, they experience a very special feeling of accomplishment.

In summary, student understanding of mathematical ideas builds through the pre-K–12 years, as a result of carefully designed experiences of active engagement with mathematics content. Across the grades, student understanding of concepts and procedural fluency grows and deepens. The kind of teaching provided to students determines the kind of learning that occurs.

The Assessment Principle

Mathematics instructional programs should include assessment to monitor, enhance, and evaluate the mathematics learning of all students and to inform teaching.

At every level, good assessment systems involve practices that enhance student learning. In 1989, the *Curriculum and Evaluation Standards* specifically addressed the purposes of assessment, including assessments designed for diagnostic purposes, instructional feedback, grading, comparison of general mathematics achievement, and program evaluation (NCTM 1989, 200–201). The original *Standards* documents call for assessment that promotes students' learning by emphasizing the thinking and reasoning of students and affording students a variety of opportunities to demonstrate what they know.

No matter what its purpose, assessment begins with knowing what is to be assessed and why the assessment is taking place. For users of this document, the "what" is students' knowledge, understanding, and use of mathematics content and processes described in the ten standards presented in chapter 3. The "why" varies with the purpose and audience for the assessment results. The closer to the classroom the stakeholders are, the more specific their assessment information needs to be. Teachers use assessment to inform their planning and decision-making, monitor individual students' progress, provide feedback that enhances student learning, and evaluate students' achievement of the mathematics goals for their grade or course. Parents also rely on assessments to provide information on how their child is performing in school. Beyond the classroom, schools and school systems, school boards, and the general public are interested in student progress toward achievement of mathematical goals. These audiences want to know if students in each grade are making appropriate progress each year. These stakeholders have an interest in the effectiveness of the mathematics instructional program as a whole.

The challenge is to employ appropriate, efficient methods for gathering sufficient information to guide decision-making without unnecessary intrusion into instructional time. In other words, what is to be assessed, who needs the assessment information, and how that information is to be used should together determine the design of the assessment.

Assessment as a Process

Assessment should be viewed as a process that involves planning, collecting information, interpreting results, and making decisions. Teachers repeat the assessment process hundreds of times as they make long-range and unit plans as well as daily lesson decisions. Daily, this process is enacted many times in a very informal way. Teachers determine the goals for the day's mathematics lesson; they ask questions and observe students during class; they interpret the information gained through these informal assessments; and they decide whether to continue or to review points now or to revisit the mathematics later. This cycle is reflected in figure 2.1.

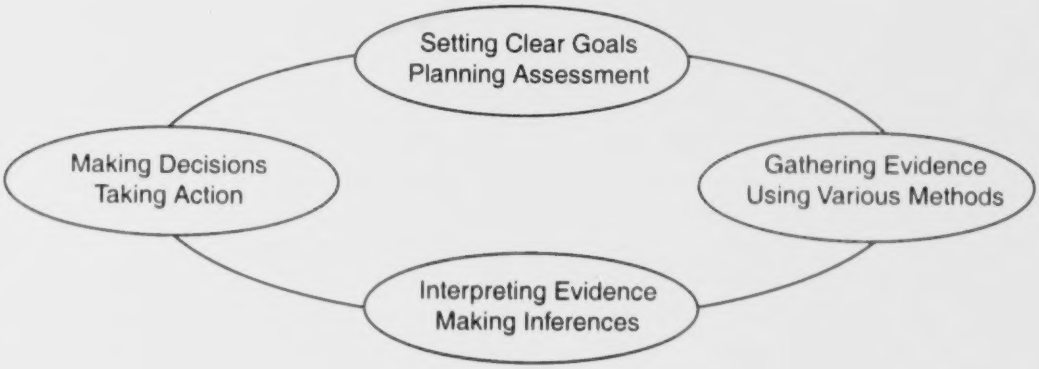


Figure 2.1. The classroom assessment cycle.

Establishing and Setting Clear Learning Goals

Students and teachers need the greatest amount of information and the most detailed evidence from assessments. Specifically, teachers need to understand what students are thinking, how they approach problems, and what content they have learned. In an instructional environment that demands a deep understanding of mathematics, assessments should be based on clear, comprehensive learning goals. The standards in this document describe mathematics goals for all students in general terms. However, it is not the intent of this document to provide detailed learning outcomes or benchmarks for specific grade levels. It is the responsibility of provincial, state, and local guidelines to provide more definition and guidance. The curriculum materials chosen to enhance classroom instruction in each grade or course should articulate learning goals to a greater degree of specificity.

In their daily planning, teachers need to focus on specific learning goals within the context of long-term goals. For example, the Geometry Standard for grades 6–8 addresses the focus area of shape, structure, and visualization by calling for students to "Understand properties of two- and three-dimensional figures and the relationships among figures." Different curriculum materials may address this goal in a variety of ways, and the instructional sequences will likely identify several more specific instructional emphases. These might include aspects of factual information (such as naming various quadrilaterals); conceptual understandings (such as knowing that a square is a special kind of rectangle); skills (such as constructing quadrilaterals with rulers, compasses, protractors, or software); and problem solving or application (such as whether a network of roads between two towns can be traversed in exactly 3 distinct ways).

Gathering Evidence

There are many important questions that are to be answered about students' progress and achievement. Specifically, to what extent do students understand the mathematics that was just introduced? How are students thinking about this concept? How flexible are students in approaching new problems? Have the students had sufficient practice on this technique? Can students apply this mathematical content to different situations? Do students have quick recall of these facts? Student learning—the result of mathematics instruction—should be assessed in ways that provide the clearest and most complete picture of what students know and are able to do.

There are many options available to teachers for assessing mathematical learning, including selected-response quizzes, student-constructed responses and open-ended questions, performance tasks, observations, conversations, and portfolios. Although all of these methods are appropriate for classroom assessment, some lend themselves more readily to some types of learning goals than others. Observations and conversations afford informal feedback, giving teachers insight into students' thinking and guiding the hundreds of decisions they make daily. Short-answer quizzes may illustrate if students can apply procedures, whereas complex performance tasks may focus on students' abilities to apply mathematics in new situations. Open-ended questions that ask students to explain how they arrived at their answers may reveal information about skills as well as conceptual understandings. Reflective journals can help teachers understand students' thinking and reasoning.

The age and experience of students should influence how information is gathered so that assessment encourages student learning rather than creates barriers for further opportunities to learn. Young children, for example, are likely to have far greater understanding than they can communicate in words or symbols. Students who lack fluency in the language of instruction may also know more mathematics than they can communicate in a written assessment.

Assessment is complex because teachers must simultaneously consider students as individuals and the class as a whole. Classroom assessment is important because it influences the direction of instruction. It also provides a way of judging collective class achievement. For example, if mathematics goals include new content or process, initial assessments are likely to be diagnostic, providing feedback that helps teachers decide instructional plans. Assessments can also focus on meanings that individual students are giving to the mathematics and the processes they can demonstrate. This type of assessment should occur frequently and provide feedback to students.

Assessments administered by states, provinces, or school districts for accountability purposes or program evaluation are not likely to draw on as many different assessment strategies as teachers employ. However, it is equally important at this level to match the assessment purpose with an appropriate method so that an assessment designed for one purpose is not used indiscriminately. In most instances, the farther away from the classroom the stakeholders are, the less specific their information needs and the more appropriate sampling becomes. Data about groups of students, especially those collected over time, reveal trends in students' performance and are usually sufficient for the decisions that must be made.

Interpreting Evidence and Making Inferences

By making appropriate inferences from assessment evidence, teachers decide what to do to foster students' learning. The teacher must next decide if valid inferences can be made about all aspects of learning. For example, a fourth-grade student might be able easily to create fractions equivalent to a given fraction by multiplying the numerator and denominator by the same number and to achieve top scores on quizzes that required students to give examples of equivalent fractions. Yet this student's teacher, wondering if the student understands the concept of equivalent fractions, could pose the following task:

> Suppose you have two circles of the same size. In one, 6/8 of the circle is shaded. In the other, 3/4 of the circle is shaded. Is the same amount shaded?

If the student responded, "No, the one with 6/8 would be more," then the teacher would discover that the student could apply an algorithm accurately, without understanding the underlying concept.

Since inferences are conclusions about students' learning based on their performances, the validity of the inferences is related to the quality of the assessment evidence and the expertise of the teacher. Teachers should look for a convergence of evidence from different sources for the decisions they must make (Stiggens 1997). This part of the assessment process requires that teachers be skilled in interpreting assessment information from multiple sources and making inferences.

Assessment is most useful when it has positive consequences for student learning. Teachers should decide on the basis of learning goals, what accomplishment looks like and what kinds of problems should be solved. After making these decisions, the teacher can then better help students make connections between new mathematics and what they already know. Feedback to students is an important consequence of assessment, because it helps them set personal learning targets and assume responsibility for their own learning. Rubrics that help teachers analyze learning tasks and describe levels of proficiency can be used to help students understand what performance is expected and how that performance will be evaluated. They provide guidance for students' self-assessment and increase the student's autonomy as they advance through the pre-K–12 years.

Making Decisions and Taking Action

The purpose of classroom assessment is to help students learn and achieve high standards, not to sort and compare them. The quality and depth of learning is determined by the decisions teachers make. Therefore, assessment should inform the teacher about students' understanding and help in choosing interventions with individual students and next steps for the whole class.

A second use of classroom assessment is to assign grades and to evaluate learning in light of both content knowledge and performance expectations. It is important for teachers, students and their families to know both about short-term progress students are making as well as where the student is in relationship to long-term goals. Evaluation is the process of judging and assigning value. It is an important element of instruction just as ongoing assessment is a component of meaningful instruction. Periodic evaluations offer opportunities to inform parents about the goals of mathematics education. These judgments must be based on multiple types of evidence. They must accommodate the age, experience, and language proficiency of the student to avoid inappropriate decisions based on assessment results. Whether they measure progress over time or performance against standards for mathematics, evaluations should communicate clear information to students and their families through conferences, portfolios, or report cards.

Some year-end or end-of-course evaluations are designed to judge student success in understanding a specific body of knowledge and their performance at desired levels of proficiency. They can be linked with the summative assessment information needed by teachers and students within the classroom and the overall performance data that school administrators and district supervisors need. These evaluations provide one piece of information that can be used with other data for making decisions.

The Technology Principle

Mathematics instructional programs should use technology to help all students understand mathematics and should prepare them to use mathematics in an increasingly technological world.

This discussion focuses on electronic technologies, including calculators, computers, micro-based laboratories, and web browsers, because the advent of such technologies is one of the most fundamental factors affecting education today. Such technologies offer the potential for students to engage in significant ways with mathematical ideas. Technology can make mathematics and its applications accessible in ways that were heretofore impossible. Students can learn more mathematics, more deeply with appropriate use of technological tools (Dunham and Dick 1994; Sheets 1993; Oosterum 1990; Rojano 1996). Like any tools, technological tools can be used well or poorly. They should not be used as replacements for basic understandings and intuitions; rather, they can and should be used to foster those understandings and intuitions. Within mathematics instructional programs, technology should be used responsibly, with the goal of enriching student learning of mathematics.

In biology courses, students learn about issues like cloning and recombinant DNA; in physics courses, students hear about quarks and strangeness. Although mathematics flourishes as a discipline—there are many new discoveries, and new fields emerging as a direct result of technology access—students often experience it as a dead discipline, a relic of the past. Through technology, this problem can be addressed. Geometric visualization is an expanding field, and computer-generated images introduce students to it in ways analogous to the ways films about subatomic particles introduce that topic in physics classes. Exploratory data analysis of large data sets is another evolving area, and with technology students can learn about the processes involved. Similarly, course units now exist that introduce students to the fundamental ideas underlying fractals and chaos. Students can work at code breaking and learn about ciphers. The door is now open for students to experience some of the wonder of modern mathematics.

Understanding Important Mathematical Ideas and Delving into Content More Deeply

At every grade level there are ways in which students can experience mathematics with technology that provide deeper and more substantial understandings than are likely to be developed without it. Students can explore new mathematical territory with simple technological tools. A young student can discover that it is possible to "count" on a handheld calculator by pressing "+1" and repeatedly pressing the "equals" key. After counting from 1 to 100 that way, it would be natural next to count by 2's to 100. Trying 3's would lead to the discovery that it is impossible to arrive at exactly 100. This raises the question of which counting numbers would allow the student to get to 100 exactly—and would give rise to a very interesting set of mathematical explorations. In the elementary grades LOGO and other computer environments provide concrete ways of introducing students to the ostensibly abstract ideas of algorithms, routines and subroutines, and variables, as well as more subtle metacognitive issues such as the need for planning. The fact that children have to tell the computer what to do, and that it does *exactly* what they tell it to do, makes the specification of algorithms concrete in a way that might otherwise be inaccessible to young children. Creating a square 10 units on a side by typing a command such as

```
repeat 4: forward 10, turn right 90
```

rather than

```
forward 10; turn right 90; forward 10; turn right 90; forward 10; turn right 90; forward 10
```

introduces the concept of iteration and the idea of routines. That the square can be called "Box10" and used as part of another figure simply by typing "Box10" introduces students to the idea of subroutines. Moreover, replacing "10" by "n" in the definition creates a mathematical function, in which a different picture is produced for each value of n. Given these visual outputs, these ideas are meaningful to the children who create them. Other parts of mathematical content are accessible earlier. Children can study graphs and tables at an earlier age when they have technological help to generate them. Prior to the existence of the technology, these ideas were not easily introduced, nor as important, for young children.

At the middle school level, computer-based laboratories allow for the collection and analysis of real-time data in ways that can make the applications of mathematics meaningful and relevant. New computational environments provide tools by which students can explore the idea of change, a topic previously reserved for calculus courses. Similarly at the secondary level, technological tools can be used to help introduce students to important ideas. For example, before there was ready access to graphing technology, it was difficult to introduce maximum-minimum problems because of the algebraic complexity involved. Students could set up the problems—an important modeling exercise—but without the techniques of calculus to find precise critical values, and the best one could do by hand was to sketch the functions and provide crude estimates of critical values. With graphing tools available, students can at least see the shape of the resulting graph and obtain an estimate of the answer empirically. Similarly students' ability to deal with mathematical applications often depended on their ability to deal with the equations that resulted—so, for example, detailed comparisons of exponential and polynomial growth were beyond reach. With symbolic manipulation tools available, that is no longer the case.

Dynamic geometry programs can enhance students' experience of two-dimensional and three-dimensional geometry (Healey 1993; Lehrer and Chazan in press). Such programs make it easy to generate a large set of instances, and to make conjectures. (e.g., "The medians of a triangle all intersect. What happens if I try the line segments that trisect the opposite sides?") In a technologically rich instructional setting, exploration and conjecturing can be a central part of geometry. Students can work on proving conjectures they have generated themselves in addition to proving known results provided as exercises. Computer algebra systems, which include graphing and symbol manipulation utilities, make explicit links between algebraic and graphical representations of functions. Students can explore how changing the coefficients of a quadratic function affects the shape of the graph, or how the graphs of $f(x+a)$, $f(ax)$, and $f(x) + a$ compare to the graph of $f(x)$. The topics of limits and convergence can be approached empirically—so that students can build intuitive foundations for some complex ideas and be helped to visualize them as well. In the content area of discrete mathematics, students can write inductive programs, specify discrete sets, or specify counting arguments as instructions for computers as a way of better understanding concepts related to algorithms (Asiala et al. 1996). Technological tools can make pursuing the general and the abstract more accessible.

Even with the potential described above, a large caveat must be issued. The enriched student understanding described in the examples above is not only possible but also likely—IF the technology is used appropriately. Flashy, show-and-tell demonstrations may leave students no more knowledgeable than before. "Black box" computations can leave students in the dark, when they should have developed intuitions about the objects being computed. Dynamic geometry programs that can "test" conjectures with ease may leave students with the impression that proof is no longer necessary, and that empirical confirmation will guarantee the correctness of a conjecture. The technology should be used to support conjecture, but teachers need to provide an emphasis on the importance of proof.

Calculators

Today's calculators provide a level of computational support for problem solving that was unimagined even when the 1989 *Standards* were released. With calculators, students can have access to a wider range of complex problems, and they can address these problems earlier in their school experience. Students at all levels should have access to calculators and other technology to use as they solve problems. Research affirms the positive impact of calculator use on the development of problem solving strategies and performance (Heid 1998; Hembree and Dessart 1985). However, access to calculators does not replace the need for students to learn and become fluent with basic arithmetic facts, to develop efficient and accurate ways to solve multidigit arithmetic problems, and to perform algebraic manipulations such as solving linear equations and simplifying expressions. Students should learn to make sensible choices, using technology in situations where the calculations are complex or numerous, but not relying on technology when answers are easily obtained or in places where they should have procedural fluency. Teachers have the responsibility to make appropriate choices about the use of calculators in particular contexts, on the basis of how well the tool will help or hinder the intended mathematical development. Technology is an important part of our world, and schools must affirm the appropriate use of calculators at all levels as a legitimate and important tool for learning and doing mathematics.

Preparing for the Increasing Demands of Society

Today, second and third graders search the World Wide Web for games, conversation, and information. Many of today's middle school students are more technologically sophisticated than were college graduates ten years ago. Today's jobs demand the use of mathematically driven technological tools. If schools do not have a level of technology equivalent to the level found outside of schools, and if they do not prepare students appropriately with it, then they are placing their students at a serious disadvantage.

There are now projects in which students, nationwide, gather and exchange data about acid rain and share and analyze those data using the World Wide Web. Students who live in different cities engage in collaborative projects. Students can communicate with mathematics mentors at a distance. They can publish their work electronically, making it accessible to others. All of this, in addition to being motivating, provides opportunities for feedback and for yet richer engagement with the mathematics. Such experiences prepare students for entry into a world in which fluency with technological tools will be expected. And, if students learn to use these tools appropriately, with understanding of their purposes, strengths, and limitations, they will be able to use the technologies much more wisely in their careers.

As the equity principle makes clear, making certain that all students have routine access to the kinds of technological tools discussed here is essential. This means having them in the classroom, so they are available when needed. It means ensuring that teachers have adequate technological and mathematical knowledge to provide appropriate support for their students. To deprive students of such opportunities is to handicap them, and that is fundamentally inequitable.

Chapter 3
Overview of Standards for Grades Pre-K–12

The guiding principles for school mathematics programs discussed in chapter 2 provide directions to be considered in instructional classrooms, schools, districts, and beyond. They are a basis for the curricular suggestions that appear in this draft of *Principles and Standards for School Mathematics*. Central to these guiding principles is the question, What mathematical content and processes should students know and be able to use as they progress through school? The ten standards presented in this chapter are intended to address that question.

Societal needs for mathematical understanding have never been greater, and they will continue to increase. These needs include the following:

> **Mathematical literacy**. The underpinnings of everyday life are increasingly mathematical and technological. Our students will live in a world where intelligent decisions often require quantitative understandings.
>
> **Cultural literacy**. Mathematics is a great cultural and intellectual achievement of humankind, and our citizens should develop an appreciation and understanding of that achievement.
>
> **Mathematics for the workplace**. Just as the level of mathematics needed for intelligent citizenship has increased dramatically, so too has the level of mathematical thinking and problem solving needed in the workplace increased.
>
> **Mathematicians, scientists, engineers and other users of mathematics**. Equity and excellence both must be the object of school mathematics programs. By enfranchising more students, while maintaining high standards, there will be a larger number available to pursue these careers.

What Are Appropriate Mathematical Goals for Students?

The standards presented here are ambitious but necessary to achieving a society that is capable of thinking and reasoning mathematically. To be productive members of that society, all citizens must develop a common base of mathematical knowledge and skill. The standards describe the knowledge base through a connected body of mathematics understandings and competencies.

The standards address both mathematical content and mathematical processes. Roughly speaking, the content standards represent what students should know; the process standards represent ways of acquiring and using that knowledge. This separation is artificial, however. In practice, what one can do depends in important ways on what one knows and on how one can exploit that knowledge.

Five standards describe the mathematical content that students should learn:

- Number and Operation
- Patterns, Functions, and Algebra
- Geometry and Spatial Sense
- Measurement
- Data Analysis, Statistics, and Probability

Five standards describe the mathematical processes through which students should acquire and use their mathematical knowledge:

- Problem Solving
- Reasoning and Proof
- Communication
- Connections
- Representation

It should be recognized that this set of ten standards does not neatly separate the content of school mathematics into nonintersecting subsets. Because mathematics as a discipline is highly interconnected, the areas overlap and are integrated. Some topics in data analysis, for example, could be characterized as part of measurement. Patterns and functions appear throughout geometry. Number pervades all the areas of mathematics. The arrangement of the standards is designed solely as a way of organizing the content and processes. Any method of dividing the content of mathematics will, at the top level, highlight some areas and obscure others. Consider discrete mathematics as an example. In this draft of *Principles and Standards*, the main topics of discrete mathematics are included, but they are distributed across the standards, rather than separated into a separate standard.

The process standards point to major aspects of student competency that are essential to students' mathematical growth. As students learn mathematics, they will develop an increasing repertoire of problem-solving skills, a wide range of mathematical "habits of mind," and increasing sophistication in mathematical argument. Also, students should become proficient at expressing themselves mathematically, both orally and in writing, gain fluency in the language of mathematics and able to make connections within mathematics and from mathematics to other disciplines.

Aiming for Focus and Coherence: Growth Across the Grades

A significant change between the *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989) and this draft of *Principles and Standards* is that there are ten standards, and each applies across the pre-K–12 grade span. Within each standard, a small number of focus areas is identified. The standards overviews that follow in this chapter are intended to describe the focus areas within each standard as they are initiated, developed, and completed across the grades. In addition, the overviews give a sense of how students' understanding and proficiency develops in each area across pre-K–12. By organizing according to common standards across the grade levels, one can chart the growth in knowledge and sophistication of students as they progress through the curriculum. Each overview contains a summary of the main points that are elaborated in the detailed descriptions of the content at each grade-band (pre-K–2, grades 3–5, grades 6–8, and grades 9–12) that are presented in chapters 4 through 7.

Readers may wish to explore this document "by strand"—that is, by following a particular standard across the pre-K–12 span—rather than progress through the standards overviews and grade-band discussions. This can be done most conveniently in the electronic version of *Principles and Standards*, available on the Web at **<http://standards-e.nctm.org>**.

#3

Reader Reaction

As you read the next sections, please consider the following:

- **What is your reaction to these ten standards as a set?**
- **Please react to the statements of the standards. Are they clearly stated?**
- **Does each overview convey a clear image of the development of the standard over the pre-K–12 span? Where is the text unclear? What kinds of examples are helpful?**
- **Do discrepancies exist between the standards overviews and the material in each grade-band chapter? Where are the discrepancies, if any?**

Mathematics Curriculum Standards:
Pre-K–12

Standard 1: Number and Operation

Mathematics instructional programs should foster the development of number and operation sense so that all students—

- understand numbers, ways of representing numbers, relationships among numbers, and number systems;
- understand the meaning of operations and how they relate to each another;
- use computational tools and strategies fluently and estimate appropriately.

Standard 2: Patterns, Functions, and Algebra

Mathematics instructional programs should include attention to patterns, functions, symbols, and models so that all students—

- understand various types of patterns and functional relationships;
- use symbolic forms to represent and analyze mathematical situations and structures;
- use mathematical models and analyze change in both real and abstract contexts.

Standard 3: Geometry and Spatial Sense

Mathematics instructional programs should include attention to geometry and spatial sense so that all students—

- analyze characteristics and properties of two- and three-dimensional geometric objects;
- select and use different representational systems, including coordinate geometry and graph theory;
- recognize the usefulness of transformations and symmetry in analyzing mathematical situations;
- use visualization and spatial reasoning to solve problems both within and outside of mathematics.

Standard 4: Measurement

Mathematics instructional programs should include attention to measurement so that all students—

- understand attributes, units, and systems of measurement;
- apply a variety of techniques, tools, and formulas for determining measurements.

Standard 5: Data Analysis, Statistics, and Probability

Mathematics instructional programs should include attention to data analysis, statistics, and probability so that all students—

- pose questions and collect, organize, and represent data to answer those questions;
- interpret data using methods of exploratory data analysis;
- develop and evaluate inferences, predictions, and arguments that are based on data;
- understand and apply basic notions of chance and probability.

Standard 6: Problem Solving

Mathematics instructional programs should focus on solving problems as part of understanding mathematics so that all students—

- build new mathematical knowledge through their work with problems;
- develop a disposition to formulate, represent, abstract, and generalize in situations within and outside mathematics;
- apply a wide variety of strategies to solve problems and adapt the strategies to new situations;
- monitor and reflect on their mathematical thinking in solving problems.

Standard 7: Reasoning and Proof

Mathematics instructional programs should focus on learning to reason and construct proofs as part of understanding mathematics so that all students—

- recognize reasoning and proof as essential and powerful parts of mathematics;
- make and investigate mathematical conjectures;
- develop and evaluate mathematical arguments and proofs;
- select and use various types of reasoning and methods of proof as appropriate.

Standard 8: Communication

Mathematics instructional programs should use communication to foster understanding of mathematics so that all students—

- organize and consolidate their mathematical thinking to communicate with others;
- express mathematical ideas coherently and clearly to peers, teachers, and others;
- extend their mathematical knowledge by considering the thinking and strategies of others;
- use the language of mathematics as a precise means of mathematical expression.

Standard 9: Connections

Mathematics instructional programs should emphasize connections to foster understanding of mathematics so that all students—

- recognize and use connections among different mathematical ideas;
- understand how mathematical ideas build on one another to produce a coherent whole;
- recognize, use, and learn about mathematics in contexts outside of mathematics.

Standard 10: Representation

Mathematics instructional programs should emphasize mathematical representations to foster understanding of mathematics so that all students—

- create and use representations to organize, record, and communicate mathematical ideas;
- develop a repertoire of mathematical representations that can be used purposefully, flexibly, and appropriately;
- use representations to model and interpret physical, social, and mathematical phenomena.

Standard 1: Number and Operation

Mathematics instructional programs should foster the development of number and operation sense so that all students—

- **understand numbers, ways of representing numbers, relationships among numbers, and number systems;**
- **understand the meaning of operations and how they relate to each other;**
- **use computational tools and strategies fluently and estimate appropriately.**

Elaboration: Pre-K–12

Number, operation, and computation have a long and prominent history in the school mathematics curriculum. In addition, this area of mathematics, perhaps more than any other, is widely-recognized and valued beyond the school setting. Central to this standard is the goal of developing number sense—understanding what numbers mean, how they relate to one another, their relative size, how they can be thought about and represented in many ways, and the effect of operating with numbers. With guidance and meaningful experiences, over time students develop "good intuition about numbers and their relationships" (Howden 1989). Students with number sense naturally decompose numbers, develop and use benchmarks as referents, use the relationships among operations and their knowledge about the base-ten number system to solve problems, estimate a reasonable result for a problem, and have a disposition to make sense of numbers, problems, and results (Sowder 1992). Students with skills embedded in a sense of number and operations are confident users of mathematics.

Knowledge of basic number facts is essential to the development of number sense and is fundamental in enabling students to solve problems. Students must be able to recall basic facts easily. These facts include the single-digit addition combinations and the counterparts for subtraction, multiplication, and division. Basic fact understanding and skill can be developed through the exploration of thinking strategies such as "7+8 is the same as 7+7+1." Understandings and skills can also be developed through varied and systematic practice sessions at home and school. Most students should be able to recall addition and subtraction facts quickly by the end of grade 2 and recall multiplication and division facts with ease and facility by the end of grade 4.

Likewise, *computational fluency*— having and using efficient and accurate methods for computing—is essential in the development of number sense and to success in most areas of mathematics. In some cases, students will use mental strategies, such as thinking of "6 x 2.5" as "six twos plus six halves." In other cases, students use a combination of mental strategies and jottings on paper to produce quick and accurate results. In still other cases, students may use paper and pencil to perform student-generated or conventional computational algorithms, particularly when the numbers are large or complicated. The main point is that students must have methods that they can use efficiently and that produce correct answers. Using and manipulating materials and reflecting on and comparing strategies helps students develop an understanding of numbers, operations, and their properties and will lead to knowledge of basic facts as well as computational fluency.

All students should acquire strategies for estimating in computational situations and the inclination to judge the reasonableness of numerical data, including computed results. Ability and inclination to estimate depends on understanding numbers—their size, position in the number system, and equivalent forms—and on the effect of operating on those numbers (e.g., "What happens when a whole number is multiplied by a number less than 1?"). Estimation can be used to answer a question directly, such as "How much pizza should we order?" or used to evaluate the reasonableness of an answer resulting from paper and pencil or calculator computation. By the time they reach high school, students should understand measurement error and its effects on computations and should develop the ability to distinguish between estimation and approximation.

Calculators are accessible and reliable tools for computing. All students should use calculators at appropriate times as computational tools. The calculator should be considered a legitimate tool within the mathematics classroom for computing, particularly when many or cumbersome computations are necessary for solving problems. However, when the instructional focus is on developing student-generated or conventional computational algorithms, the calculator should be set aside to allow for this focus. Today, the calculator is a commonly used computational tool outside the classroom. The environment inside the classroom should reflect this reality.

- **Understand numbers, ways of representing numbers, relationships among numbers, and number systems**

Students' knowledge of number concepts and their properties should grow more sophisticated throughout their school experience. In grades pre-K–2, students learn to count, represent, and compare the size of numbers in a variety of ways, aided by physical representations that they can manipulate, such as counters and base-ten blocks. Students in grades pre-K–2 will encounter and should explore numbers larger than they can manipulate physically. In fact, they are often very interested in large numbers, particularly if embedded in contexts from their own experiences. For example, young students can count and keep an ongoing record of the number of coins collected in the school penny drive or the number of soda-can tabs collected in a conservation project. Students in the early grades explore and use part-part-whole relationships. In this way, "24" is seen as "two tens and four ones"; it is also "two sets of twelve." Viewing numbers in flexible ways provides a basis for understanding the way numbers are represented within the base-ten number system.

By grade 2, students can make a transition from viewing a "ten" as simply the accumulation of "ten ones" to seeing it both as "ten ones" *and* as "one ten." This recognition is a first step toward understanding the structure of the base-ten number system (Cobb and Wheatley 1998). In the early grades, students can also encounter and learn about common fractions (e.g., 1/2, 3/4) prompted by meaningful contexts and building on existing language. For example, most young students already use the word *half* in their vocabulary outside of school.

Students in grades 3–5 continue to develop and extend their ideas and range of strategies for thinking about and using whole numbers. The knowledge that "3408" is "three thousands, four hundreds, and eight ones" is a foundation for students' understandings of how "3408" is related to "4408," to "3308," and to "3500." Such understandings are part of the ongoing development of number sense and contribute to generating and using strategies for computing.

At grades 3–5, students will study and represent fractions and decimals as numbers, with emphasis on how they are related to whole numbers and to one another. Understanding how a fraction or decimal part of a quantity is related to the unit is a key idea in these grades. Instructional emphasis at this grade band should be on the development of rational number concepts rather than on rational number computational strategies. Useful experiences for students in grades 3–5 include creating physical representations of fractions and decimals, comparing fractions to familiar benchmarks such as 1/2, representing fractions and decimals on a number line, and generating equivalent representations of fractions and decimals. With these understandings, students should be able to estimate fraction sums. For instance, "1/2 + 3/8" must be less than 1 because each addend is less than or equal to 1/2.

As students in grades 3–5 study numbers, they also learn about classes of numbers and their characteristics, such as which numbers are odd, even, prime, composite, or square. Recognizing such characteristics lays a foundation for checking divisibility rules, finding prime factors, or understanding functional relationships.

In grades 6–8, students expand their work with fractions, decimals, and percents, so that they are able to move flexibly among equivalents and apply a range of strategies to order and compare rational numbers. The shift in thinking of fractions as "parts of" specific units to understanding fractions as numbers is completed in the middle grades. Students' knowledge about, and use of, decimals in the base-ten system also extends in these grades. In addition to estimating with rational numbers , students in grades 6–8 also develop computational strategies for fractions and decimals.

The understanding of very large numbers and what they represent continues to develop through the middle grades. Students use tools such as calculators and spreadsheets to organize and analyze numbers in applied contexts, and they learn to represent very large and very small numbers in scientific notation. By extending from whole numbers to integers, middle-grade students' intuitions about order and magnitude expand. They also encounter irrational numbers, such as $\sqrt{2}$ and π, in the course of studying the Pythagorean Theorem and circumference.

Other areas of the curriculum are more prominent than number in grades 9–12; nevertheless, students continue to deepen their understanding of properties of numbers with which they are already familiar and see number systems from a more global perspective. Scientific notation and matrix representation are possible. Complex numbers are also added to students' repertoire, and students learn that not all properties of real numbers are preserved when the system is enlarged.

- **Understand the meaning of operations and how they relate to each other**

As they develop computational fluency, students must understand the meaning of arithmetic operations. This includes deciding which operations should be used for a particular problem, how the same operation can be applied to problem situations that appear to be quite different from one another, how operations relate to one another, and what kind of result to expect.

During the primary grades, students encounter a variety of meanings for addition and subtraction. Subtraction can be viewed through a "take away" interpretation or as a comparison of two sets. Missing-addend situations highlight the relationship between addition and subtraction. Multiplication and division can begin to have meaning for students in grades pre-K–2 as they solve problems that arise within their environment. Such problems might include these: How many slices of bread are needed to make four sandwiches? How can this bag of raisins be shared fairly among four people? Although the major instructional emphasis in pre-K–2 is on addition and subtraction, students who naturally encounter and are curious about situations involving other operations, such as multiplication and division, should be encouraged.

Focusing on the meaning of multiplication and division, particularly with whole numbers, becomes central in grades 3–5. By creating representations of multiplication and division situations, either with diagrams or concrete objects, students gain a sense of the relationships among addition, subtraction, multiplication, and division. By inventing strategies for computing as well as by using conventional computational strategies, students use and recognize such properties as associativity, commutativity, and distributivity and recognize 0 and 1 as identities. The development and comparison of computational strategies provides a chance to focus on the nature of algorithms. For example, many multiplication algorithms illustrate distributivity, and division strategies sometimes involve an iterative process.

Understanding operations with rational numbers is emphasized in grades 6–8. Students at this level also represent and develop operations on integers. Students' intuitions about operations need to be revised with movement into an expanded number system (Graeber and Campbell 1993). For example, the multiplication of a positive number by a fraction smaller than 1 produces a result smaller than the number, counter to students' prior experience that multiplication always results in a bigger number. And, working with integers, subtraction no longer "makes smaller". Students also should focus on the inverse relationship between multiplication and division and the relationship between a fraction and its reciprocal. Proportional reasoning is fundamental for operating with fractions, decimals, rates, and ratios in grades 6–8. Prior to these grades, most of the comparisons students have done are additive, such as "how much taller?" or "how many more?" In the middle grades, students should become proficient in creating ratios to make comparisons in situations that involve pairs of numbers, as in the problem, "If three packages of cocoa make fifteen cups of hot chocolate, how many packages are needed to make sixty cups?"

In grades 9–12, students continue to study operations and establish connections between the operations and other topics. Addition of complex numbers is equivalent to addition of vectors, whereas multiplication by complex numbers has the geometrical interpretation of a rotation combined with stretching (or shrinking). Functional operations—such as finding *n*th roots, absolute values, and raising to powers—build on earlier work and familiarity with number. Properties of operations, such as closure, are part of understanding algebraic systems.

- **Use computational tools and strategies fluently and estimate appropriately**

A variety of efficient computational tools exist and are used regularly by adults, including mental computation, paper and pencil strategies, estimation, and calculators. Students need experiences that help them choose among tools appropriately. The particular context, question, and numbers involved should be considered when choosing a method. Do the numbers allow a mental strategy? Does the context call for an estimate? Does the problem require repeated and tedious computations? Students should evaluate problem situations to determine whether an estimate or an exact answer is needed and be able to give a rationale for their decision. Estimation strategies and exact computation strategies should be used in tandem as students solve problems.

From pre-K to grade 8, students are expected to develop computational strategies based on their knowledge about numbers and operations and to explain and justify their procedures. By describing the algorithms they use to the teacher and their peers, students see that multiple solution procedures are possible and that some are more efficient than others. Students should become computationally fluent—that is, possess efficient and accurate methods for solving computational problems. Developing fluency requires a balance and connection between conceptual understanding and procedural computation and relies on quick access to basic number facts. On the one hand, computational strategies practiced and applied without a conceptual base are often forgotten or remembered incorrectly (Kamii 1998). On the other hand, understanding without fluency can inhibit the problem-solving process.

As children in grades pre-K–2 develop understanding of whole numbers and the operations of addition and subtraction, instructional attention should focus on developing strategies for computing with numbers of varying size, such as single-digit and multidigit whole numbers. Student-generated strategies should be shared and discussed. By the end of grade 2, students should be able to recall basic addition and subtraction facts, should be fluent in adding two-digit numbers, and should have methods for subtracting two-digit numbers.

In grades 3–5, both student-generated and conventional computational strategies for adding, subtracting, multiplying, and dividing whole numbers are studied, applied to larger numbers, and practiced for fluency. Based on his view of relevant research, Gravemeijer (1998) notes:

> The underlying idea is that the students develop mathematical concepts, notations, and procedures as organizing tools when solving problems. In such a process, informal algorithms may come to the fore as forms of well-organized routines for solving certain types of problems. With guidance of the teacher, these informal algorithms can be developed into conventional algorithms. However, one may also opt for fostering the informal algorithms, which are valuable as end goals as well. (p. 4)

Students also develop and begin to apply computational strategies to decimals. In these grades, students develop recall of basic multiplication and division facts. Rational number concepts are a major instructional goal at this level, and these concepts lead to informal methods for calculating with fractions. For example, in grade 5, a problem such as "1/4 + 1/2" should be solved mentally with ease, because students should have clear geometric images of 1/2 and 1/4 or be able to use decomposition strategies, such as "1/4 + 1/2 = 1/4 + (1/4 + 1/4)."

Fraction and decimal computation is an instructional focus in grades 6–8. Strategies for computing with fractions should build on conceptual knowledge developed in earlier grades. Students should come to grades 6–8 able to compute with commonly encountered fractions based on mental and concrete representations. In grades 6–8, students should develop more general computational strategies that can be applied to a full range of fraction situations. They also should extend the strategies for computing with whole numbers to decimal numbers. In grades 6–8, students are expected to become fluent in computing with rational numbers. As they develop understanding of the meaning and representation of integers, they should also develop ways of computing with integers.

In grades 9–12, students should analyze and compare algorithms as part of studying the role of algorithms in mathematics. By comparing algorithms, they consider which are easy to explain, easy to use, and are most efficient. They should be able to read a flow chart and decide if it describes a correct method for determining whether a number is divisible by 3. Students analyze why and how algorithms are constructed. At this level, students can study common algorithms for computing with whole and rational numbers as well as unfamiliar algorithms that they encounter for the first time in high school, including, for example, algorithms for finding real roots or finite differences of sequences.

Standard 2: Patterns, Functions, and Algebra

Mathematics instructional programs should include attention to patterns, functions, symbols, and models so that all students—

- **understand various types of patterns and functional relationships;**
- **use symbolic forms to represent and analyze mathematical situations and structures;**
- **use mathematical models and analyze change in both real and abstract contexts.**

Elaboration: Pre-K–12

Patterns, functions, and algebra encompass the systematic use of symbols, algebraic characteristics of mathematical systems, modeling of phenomena, and the mathematical study of change. These notions are not only linked to one another, but also closely linked to number and operations and to geometry. They are essential to all areas of mathematics and form the basic language in which mathematics is expressed. Ideas included within this standard compose a major component of the school curriculum.

Algebra has its roots in the study of solutions to equations. The subject has developed in several directions, including the study of functions, reasoning about abstract objects, generalizations, and the centrality of symbolic notation. All of these developments should be reflected in the school curriculum.

The study of patterns, functions, and algebra should begin informally in the early grades and develop in sophistication and breadth over the years of schooling. Early experiences with concepts of patterns, function, and algebra can provide a substantial base of understanding to prepare students for the more intense focus on this area in late middle school and throughout secondary school (Smith 1998).

- **Understand various types of patterns and functional relationships**

Making, recognizing, and extending patterns is a natural activity for young children. Early work with patterns is concerned with identifying a regularity, recognizing the same pattern in different forms, and using patterns to predict values. For example, "red-blue-blue-red-blue-blue-red-blue-blue..." is the same as "ABBABBABB...," and the twelfth element will be blue.

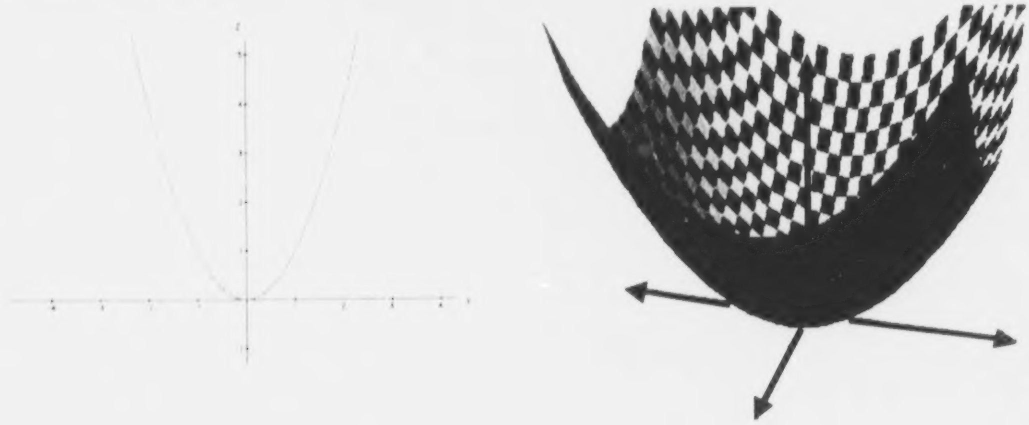
Patterns emerging from simple situations are precursors of both function and sequence. For instance, if a toy costs $2, how much do I spend if I buy 1 toy? 2 toys? 3 toys? *n* toys? Subsequent work with both growing patterns, such as "1, 3, 6, 10, 15, ...," and repeating patterns, such as "1, 1, 3, 1, 1, 3, ...," lays further foundation for these notions. In the middle grades and secondary school, the regularities underlying patterns and sequences become more complex, including those found in exponential growth. Work with sequences as examples of functions is extended in the secondary school in order to establish the foundation for the notions of limit and of infinite series.

In early grades, students can describe patterns like "2, 4, 6, 8, ..." by focusing on how a term is obtained from the previous number—in this case, by adding 2. This is the beginning of recursive thinking. Later, students can study sequences that can be defined and computed best by using recursion, such as the Fibonacci sequence, "1, 1, 2, 3, 5, 8, ...," in which each term is the sum of the previous two terms. Recursive sequences appear naturally in many disciplines and can be studied using technology. Students in grades 9–12 can study functions and patterns that are recursively generated.

As an important step in their initial work with patterns, students often describe the underlying regularity verbally rather than with mathematical symbols (English and Warren 1998). One goal of school mathematics instruction is to build on this verbalization and provide students with sufficient experience to become comfortable with, and fluent in, using mathematical symbols to represent generalizations.

Early precursors to functions and their representations include such activities as recording daily temperature or changes in the height of a plant over time on graphs. Graphical representations of functions can be used as students move through the elementary grades. In grades 6–8, linear functions and interpretation of graphs are particularly important objects of study. Interpretation of graphs retains its importance as students in grades 9–12 systematically study other families of functions, including polynomial, rational, exponential, and trigonometric functions. This systematic study in the secondary school should build on earlier school experiences with algebraic ideas.

Familiarity with symbolic, numerical, and graphical representations of functions is important, as is the ability to move thoughtfully and easily among these representations. Coordinate geometry makes possible the graphical representation of functions and relations and the investigation of their geometric properties, such as symmetries of the graph. Graphing calculators and computers assist students as they experiment with graphical and numerical representations and examine and compare different classes of functions. Relations between functions involving two and three variables can have geometric descriptions; for example, what happens when the parabola $z = y^2$ (in the y–z plane) shown in figure 3.1(a) is rotated about the z-axis? How can the new shape be described algebraically?



(a) (b)

Figure 3.1. Rotated parabola.

Many students first come to understand the notion of function as a set of instructions, such as "given n, find 2^n for $n = 0, 1, 2$, and 3" (Vinner and Dreyfus 1989). To help students develop more advanced understanding of functions, considerable experience with multiple representations of functions—including numeric, graphic, and symbolic—is required.

- **Use symbolic forms to represent and analyze mathematical situations and structures**

Symbolic representation of quantitative relationships lies at the heart of algebra. Its power of compression allows complex mathematical ideas to be expressed succinctly, and the symbols and expressions facilitate exploration and solution finding. However, along with this power comes a set of conceptual obstacles. For example, the notion of variable is quite complex. In the elementary grades, students typically develop a notion of variable as a placeholder for a specific number, as in

$$\square + 2 = 11.$$

Later, students learn that the variable x in the equation $3x + 2 = 11$ has a very different use from the variable x in the equation $(x + y)/2 = \sqrt{xy}$, and both uses are quite different from the use of r in the formula $A = \pi r^2$. A thorough understanding of variable develops over a long period of time, and it needs to be grounded in extensive experience (Wagner and Parker 1993).

Another example of conceptual difficulty in understanding symbolic representation of quantitative relationships pertains to the notion of equality. The equal sign can be perceived in quite different ways. For example, as a consequence of extensive experience with the use of the equal sign in arithmetic computation, students typically perceive the equal sign operationally, that is, as a signal to compute (Kieran 1981). Before secondary school, however, students also need to learn to view the equal sign as a symbol of equivalence and balance. In general, if students are asked to engage extensively with symbolic manipulation before they develop a solid conceptual foundation for their work, they will be unable to do more than mechanical manipulations (Wagner and Parker 1993). The foundation for meaningful work with symbolic notation needs to be laid over a long period of time, beginning in the elementary grades, well before middle or secondary school when they may see curriculum formally labeled "algebra."

When young children work with number, they often adopt personal strategies that are algebraic in nature. Teachers can build on these natural tendencies in appropriate ways. For example, a child might notice that "4 + 5 = 4 + 4 + 1," and "5 + 6 = 5 + 5 + 1," and so on. Explaining her or his observation to another child, the student might draw the picture shown in figure 3.2.

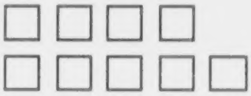


Figure 3.2. Double plus 1.

The use of a picture as a paradigm and not as a record of an isolated event makes the pictorial representation algebraic. Or, the child might simply say, "double plus 1." Because this statement expresses a generalization, it is algebraic.

In grades 6–8, representation becomes more formal, thus adding a powerful perspective to the interplay among symbolic, pictorial, concrete, and geometric representations. Even complex algebraic relationships may become clear when they are viewed geometrically. These geometric arguments can often be understood long before students could reasonably be expected to perform sophisticated manipulations of algebraic symbols. For example, the picture in figure 3.3 helps explain why the sum of the first n odd numbers is n^2.

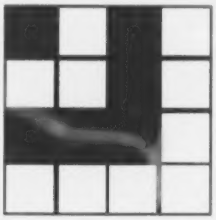


Figure 3.3. The sum of odd numbers is a square.

Students can symbolize this relationship as “$1 + 3 + \ldots + (2n - 1) = n^2$” and, later, can give a proof by mathematical induction. Thus, this algebraic generalization can be approached and demonstrated in two very different ways, one accessible to students in the middle grades and the other requiring more mathematical preparation. The two methods complement each other, and, in fact, each reveals something different about the mathematical situation.

The relation between algebra and geometry goes the other way too, as students express geometric ideas algebraically. For example, if a ball of modeling clay with a radius of r cm is reworked into a cone with the same radius, what is the height of the cone?

The notion of algebraic structure comes from noticing properties of number operations. Understanding closure properties (e.g., knowing that the sum of two whole numbers is a whole number and that “3 – 8” is not a whole number) and algebraic properties (e.g., knowing that addition is commutative and that subtraction is not) becomes important to the study of a variety of systems, including number systems, systems of polynomials, functions, and matrices. Students can reason about operations; they discover, for instance, that subtraction is the inverse operation of addition. Thinking about a complex mathematical system, asking coherent questions about it, and finding methods to answer those questions all are important in learning mathematics.

Another important component of structure is the notion of isomorphism—identical mathematical structure lying behind phenomena that could seem different. For example, two situations may have different physical manifestations and yet be modeled by the same directed graph. This would show that the two procedures share important mathematical characteristics.

- **Use mathematical models and analyze change in both real and abstract contexts**

One of the most powerful uses of mathematics is the mathematical modeling of phenomena. Using of symbolic notation is central to modeling. For example, distribution and communication networks, laws of physics, population models, and statistics for a data set all can be expressed in symbolic language. Algebra is implicit in any moderately sophisticated use of spreadsheets. If the relations among a set of numerical categories are well understood, this understanding will be expressed in the language of variables, functions, and relations.

Because of this fact, it is important for students to model a wide variety of phenomena mathematically, beginning in the early grades. As students become comfortable with standard families of functions, they can use them to model situations and identify them as linear, exponential, and so forth. Trigonometric functions are useful to represent periodic phenomena. The use of computer-based laboratories enables students to get reliable numerical data quickly from physical experiments, thus enlarging the class of situations that can be modeled. Graphical, numerical, and symbolic capabilities of computers or calculators can then be used to explore the effects of possible variations in the models. Maxima and minima become important, as do situations involving change.

The study of change eventually leads to calculus, but students can discuss change long before they might take a formal calculus course. In grades pre-K–2, students who work on a problem graphing the time versus distance for a runner are able to point out that for a while the distance increases quickly, and then the increase slows down. Any process dependent on time graphed as $y = f(t)$ has a steep slope when it changes quickly and a shallow slope when it changes slowly.

Arithmetic and geometric sequences differ in the way an element is defined in terms of its predecessor. The study of change can be connected to ideas of recursion. Elementary students can observe in patterns like 5, 8, 11, 14, … that each number is just 3 more than the last number. As students progress through the grades, they will study more complex changes in sequences such as 1, 3, 6, 10, … in which the change from one term to the next is increasing by 1. Proportional growth, such as 2, 4, 8, 16, …, in which each number is a multiple of the previous number, results in an exponential relationship, $y = 2^n$.

Overall, the concepts and skills in the areas of pattern, function, and algebra grow in complexity over the years. Similarly, students' thinking in these areas needs to evolve and mature as they progress through school.

Standard 3: Geometry and Spatial Sense

Mathematics instructional programs should include attention to geometry and spatial sense so that all students—

- **analyze characteristics and properties of two- and three-dimensional geometric objects;**
- **select and use different representational systems, including coordinate geometry and graph theory;**
- **recognize the usefulness of transformations and symmetry in analyzing mathematical situations;**
- **use visualization and spatial reasoning to solve problems both within and outside of mathematics.**

Elaboration: Pre-K–12

Much of geometry should be learned through activity, with physical models, drawings, and dynamic software as learning tools. Well-designed activities, availability of appropriate tools, and teacher support can enable students to make conjectures about geometric structure, explore the conjectures of others, and reason carefully about geometric ideas. The eventual goal is for students to systematically study geometric shape and structure and increasingly use formal reasoning and proof in their study. Geometry and spatial sense are fundamental components of mathematics education. They offer ways to interpret and reflect on our physical environment through abstraction. They can serve as tools for the study of other topics in mathematics and science. They support creative thought in all mathematics.

The utility of geometric ideas in representing and solving problems in other areas of mathematics and in nonmathematical contexts should be constant themes in students' experiences with geometry. Geometric representations can help students make sense of area and fractions; histograms and scatterplots can lead to insights about data; and coordinate graphs can be used to analyze and understand functions. Spatial reasoning is helpful in using maps, planning routes, designing floor plans, and creating art. Geometry and spatial sense can also help students see the structure and symmetry around them

- **Analyze characteristics and properties of two- and three-dimensional geometric objects**

From early encounters with their world, children begin to gain experience with shapes and the structures of space. Young children should begin to explore, identify, and describe a variety of shapes and make observations resulting from their explorations. For example, students in grades pre-K–2 may use various shapes to tile an area recognizing that rectangles work well because they have four "perfect corners." In later grades, students describe components of shapes, such as sides and angles, and the properties of shapes. For example, using physical objects or geometric software to experiment with a variety of rectangles, students in grades 3–5 could conjecture that rectangles have properties such as two pairs of equal sides, congruent and bisecting diagonals, and so forth. By grades 6–8, students should reason deductively that a subset of these properties can characterize rectangles. In grades 9–12, students should be able to use deductive reasoning and geometric axioms and theorems to verify their conjectures about shapes and to solve problems about geometric figures using formal reasoning. At all levels, students should be encouraged to provide appropriate explanations for their conjectures and solutions.

Competencies such as visualizing, describing, representing, classifying, transforming, and investigating begin to be developed with objects that can be seen, manipulated, and held. Technology enables students to have an interactive experience with a large variety of two and three-dimensional shapes. These competencies continue to develop more abstractly with the study of attributes of shapes and the relationships among properties. Eventually, students are able to describe, represent, classify, and investigate connections among relationships, expressed in logical chains within a geometric system. Students should also become increasingly capable of establishing theorems within an axiomatic system; recognizing the differences among undefined terms, definitions, axioms, and theorems; and constructing original proofs.

- **Select and use different representational systems, including coordinate geometry and graph theory**

The rectangular coordinate system is a powerful mathematical tool that allows the translation of a problem in one setting, where its solution is difficult, to a different setting, where its solution might be trivial. Knowing about rectangular coordinates is very useful for solving a wide array of problems. In particular, the system allows for the representation of location, direction, and distance. It is a bridge for linking the worlds of algebra and geometry.

Young children first learn concepts of relative position, such as above, behind, near, and between. Later they can make and use rectangular grids to locate objects in a room or locate entries in a table. In the middle and secondary grades, the coordinate plane becomes a tool for naming points. Finding distances between points in the plane by using scale on maps or the Pythagorean relationship is an important transition in the middle grades. Geometric figures can be represented analytically by using coordinates to name vertices or by forming designated shapes by choosing appropriate vertices. Geometric software, geoboards, graphing calculators, and graph paper can support students as they develop an understanding of navigation in a plane.

Students should gain experience using a variety of visual and coordinate representations to analyze problems and study mathematics. In elementary grades, for example, the number line provides a way to demonstrate the meaning of whole number addition that can then be extended to operations on other types of numbers. In grades 3–5, grids and arrays can help students understand multiplication. More complex problems can be considered in the middle and high school. In trying to minimize the distance an ambulance would have to travel to reach a new hospital from any location in the community, for example, students in middle grades might use taxicab geometry. To find the shortest airplane route between distant cities, students in grades 9–12 would use spherical geometry. And, if students were trying to minimize the cost to make a trip to several cities by airplane, they might use finite graph theory.

- **Recognize the usefulness of transformations and symmetry in analyzing mathematical situations**

Transformations are important aspects of geometric thinking. Young children come to school with intuitions not only about shapes but also about how shapes might move. Through school experiences with informal motions such as "slide," "flip," and "turn" made by using mirrors, paper folding, and tracing, elementary school students can see these ideas as mathematical in nature. At higher grades, students' knowledge about transformations becomes more formal and systematized. At grades 3–5 students should investigate the effects of transformations and be able to describe them in mathematical terms. Using dynamic software, students will begin to develop an awareness of the attributes needed to define a transformation. For example, to transform a figure using a rotation, students need to define the center of rotation, the direction of the rotation, and the angle of rotation, as shown in figure 3.4.

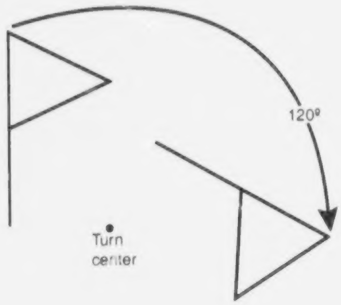


Figure 3.4. A clockwise rotation of 120°.

In the middle grades, students should develop an understanding of congruence transformations, where congruent figures coincide using a sequence of isometries—that is, transformations that preserve distance. Students should extend their knowledge of transformations to contractions and dilations and be able to describe transformations quantitatively.

Transformations should be an important tool for students in grades 9–12 to solve geometric and nongeometric problems. For example, they can be used in the study of congruence and similarity. A systematic study of the compositions of transformations can give high school students important geometric insights into algebraic properties of sets whose elements are functions. Students will be able to construct proofs concerning properties of transformations and use transformations to construct proofs in other areas.

- **Use visualization and spatial reasoning to solve problems both within and outside of mathematics**

Spatial visualization includes building and manipulating mental representations of two- and three-dimensional objects and perceiving the same object from different perspectives. One aspect of spatial visualization involves moving between two-dimensional and three-dimensional shapes and their representations. Elementary school students can wrap blocks in paper nets as a step toward learning to predict whether certain nets match certain solids. By the middle grades, students should be able to make perspective drawings and create top or side views.

Suppose, for example, that a block structure has the side view and front view shown in figure 3.5. Young children could be challenged to build a structure that satisfies either condition. At grades 3–5, students can determine if it is possible to build more than one structure satisfying both conditions. Middle and secondary school students can be asked to find the minimum number of blocks needed to build the structure.

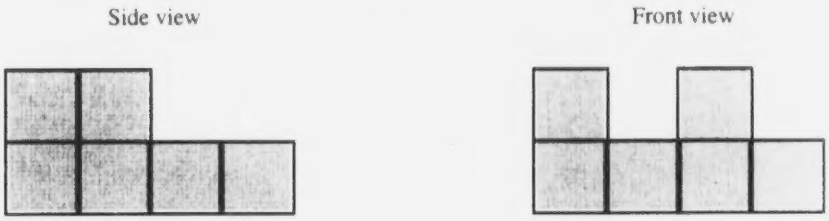


Figure 3.5. Block structure.

Students can develop visualization skills through hands-on experiences with a variety of geometric objects and through the use of technology that allows them to turn, shrink, and deform two- and three-dimensional objects.

Over the grades, students should become comfortable analyzing and drawing perspective views, counting component parts, and describing attributes that cannot be seen but can be inferred. Students need to learn to physically and mentally change the position, orientation, and size of objects in systematic ways as they develop their understandings about congruence, similarity, and transformations.

Visualization can be used as a tool in developing a conjecture or an argument. Young children might be convinced that a "diamond" is actually a square if they see it rotate to an orientation that is familiar. Older students might use spatial reasoning to decide the appropriate correspondence to use in proving that two triangles are congruent. At more advanced levels, spatial reasoning might be helpful for students in comparing volumes of solids swept out by plane regions revolving around given axes. Dilations and similarity may help students visualize proportional relationships.

Visualization and spatial reasoning are facilitated for students whose everyday experience includes extensive interaction with computer animations and other technological settings. By connecting this experience to school geometry, students can acquire important tools for problem solving in geometry and other areas of mathematics.

Throughout students' development, geometric and spatial understanding not only grows but changes in structure. Although the focus areas identified in this standard should be addressed at every grade level, the set of geometric objects that students understand and can work with will expand as they progress through school.

Standard 4: Measurement

Mathematics instructional programs should include attention to measurement so that all students—

- understand attributes, units, and systems of measurement;
- apply a variety of techniques, tools, and formulas for determining measurements.

Elaboration: Pre-K–12

Measurement is important in the pre-K–12 curriculum because of its practicality and pervasiveness in so many aspects of everyday life. The study of measurement also provides an opportunity for learning and applying other mathematics, including number operations, geometric ideas, statistical concepts, and notions of function. Many features are entailed in measurement. To most adults, these might not seem as complex as they are for students. Features include recognizing that objects have attributes that are measurable, such as length, weight, and area; selecting appropriate units of measurement; and understanding the features of measurement systems so that generalization and extension are possible. In addition, measurement can be accomplished by various techniques as well as through the use of tools, formulas, indirect measurements, successive approximations, and scaling.

- **Understand attributes, units, and systems of measurement**

A measurable attribute is a characteristic of an object that can be quantified. Line segments have length; plane regions have area; and physical objects have mass. To young children, certain attributes of objects are more salient than others. For example, a pencil's length is perceptually more obvious than its circumference or its volume. Helping young children to understand the attributes of objects is a key step in helping them learn to measure. Measurement begins as young learners compare and classify objects using comparative language such as "longer" and "shorter." Children's attention can be drawn to the attributes to be measured in a number of ways. One way, for example, is to compare the lengths of two objects by asking the question, "What can you find in the classroom that is shorter than you?" Another way is to ask children to see if a region, such as a tracing of their hand, can be covered with beans. More abstract attributes, such as volume, temperature, or angle are less easy to perceive and are more appropriate for instructional emphasis in grades 3–5 and beyond.

In the middle and high school grades, derived attributes (ratios of measurements) such as speed, density, trigonometric ratios, or variance can become part of the students' repertoire. Whatever the grade level, students should have many informal experiences in understanding attributes before using tools to measure them or relying on formulas to compute measurements.

As students gain an understanding of different attributes, they should have experience in choosing units to measure those attributes. Young children compare the object to the unit, often by iterating or using the unit to "cover" the attribute. For example, children at grades pre-K–2 may cover the length of a pencil by moving a paper clip, or they may cover a region with copies of a square they have chosen for the unit. To measure volume, they may fill a tank with glasses of water. Young children can use a wide variety of objects, such as paper clips or straws to measure length, square tiles to measure area, and paper cups to measure volume. Their measurements can be reported in terms of these units. "Standardization" of units becomes important later in the early grades, as students notice that using Joey's foot to measure the length of the classroom will give quite a different length from that found by using Adriana's foot and that therefore it is important to decide whose foot will serve as the unit. Students at this level also can begin to look for objects with a given measurement, such as a container that can be filled with twelve cups of sand or a crayon that is three paper clips long. During the middle school years, students can create a scale drawing of a room and its furnishings. The units that are part of conventional measurement systems, such as inches and feet (in the customary English system) or centimeters and cubic centimeters (in the metric system), can be used by students after they have used common materials as units.

A fundamental understanding is that the measurement will depend on the size of the unit. Toward the end of the elementary years, students should have a reasonable understanding of the role of units in measurement. By secondary school, as students are measuring attributes that are quite abstract, units become more complex, including miles per hour, pounds per square inch, and person-days. The "algebra of units" (dimensional analysis) becomes part of the focus at grades 9–12, as students learn that units in measurements behave like variables under algebraic procedures.

Learning how to choose an appropriate unit is a major part of understanding measurement. For example, it is important to realize that when measuring area, one needs to use a unit of area such as a square region. Convenience is also a consideration. Using paper clips to measure a pencil is reasonable, but not to measure the length of the soccer field. The unit chosen should produce a measurement that is a reasonable number. Precision is also important, and smaller units can lead to greater precision in the measurement. Precision involves both the choice of unit and the choice of tool; these choices are made while considering attributes of the object being measured. In grades 9–12, the notions of attribute and unit selection become more sophisticated. Defining the space within which a problem situation exists, and then choosing a structure to represent and measure of portions of the space, is the focus.

In the United States, given that the customary English system of measurement is still prevalent, students from elementary grades through high school learn both customary and metric systems. Students should understand both systems, make conversions easily within systems, and estimate measurements fluently in both. As an example, a student might say, "I live a mile from the school. That is about two kilometers." Students will find it helpful to know a few English-metric equivalents.

An understanding of measurement systems can reinforce and help students understand aspects of the base-ten system and employ proportional reasoning. For example, because of the way in which the metric system is built on a base-ten structure, consistent

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student work in metric measurement and units is likely to help with their understanding of place value. The metric system has a beautiful internal organization; the size of the next larger unit is always related in the same way to the previous unit. For example, a centimeter is ten times larger than a millimeter, a decimeter is ten times larger than a centimeter, and so forth.

There are some fundamental properties about measurement that need to be explored and understood by students at all levels. Many types of measurement are preserved by congruence—young children can find that a unit tile fits precisely on top of another and determine that their areas are the same. By the upper elementary grades, students should understand that congruent triangles, even with different orientations, have equal perimeters and equal areas. Area measurement is preserved when a shape is decomposed, or broken into pieces. Children in the elementary grades need to see that a rectangle can be broken into two triangles and that even if the parts are moved around and a new figure is formed, its area will be the same. They also can note that the same is not true of perimeter—the new figure may have a different perimeter from the original one. Such observations can provide interesting contexts for mathematical reasoning and give glimpses of sophisticated mathematical concepts such as invariance under certain transformations.

- **Apply a variety of techniques, tools, and formulas for determining measurements**

The process of measuring involves selecting a unit, comparing that unit to the attribute being measured, and arriving at a number or range of numbers. As students are developing a sense of the meaning of measurable attributes and of the ways in which units are important in measurement, they can apply techniques, tools, and formulas to find measurements. Measurement techniques are strategies used to determine a measurement. Techniques could be counting, iteration, estimation, or using tools or formulas. Measuring tools include such things as rulers, measuring tapes, vessels, scales, clocks, or stopwatches. Formulas are general relationships that produce measurements given values for the variables in the formula.

Techniques that students use in measurement vary depending on the attribute and the purpose of the measurement. For instance, consider different ways to determine the area of a rectangular floor. Students might use their prior experience with size benchmarks to estimate the area, count tiles on the floor to approximate the area based on the area of a tile, or measure the length and width and use a formula to find the area. These techniques would give roughly equal measurements, after conversion to a common unit if necessary.

Another important technique that can be used in measurement is successive approximations. Since all physical measurements are approximations, students should be encouraged to consider the bounds between which their results lie. For instance, to determine the area of a footprint, students could superimpose a transparent square grid on the drawing of the foot. Counting the squares that lie inside the boundary of the shape gives a lower bound for the area; counting the interior squares as well as those cut by the

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boundary gives an upper bound. Students should recognize that the area falls between the upper and lower bounds. Students should recognize that using grids with smaller subdivisions results in bringing the lower and upper bounds closer together. In practice, there is a limit to the number of subdivisions that can be used. This use of successive approximation is an important precursor to calculus concepts.

Measurement tools are the familiar devices that most people associate with taking measurements. Using measurement tools often relies on other mathematical knowledge such as fractions (as in determining the meaning of unlabeled tick marks on a ruler) or the ordering of numbers. Increasingly, measurement tools are technological and, therefore, mask the connection between the tool and the attribute. Consider, for example, that rulers are tools that are "close to" the length attribute by their very appearance. In contrast, dynamic geometry software used to measure an angle offers less insight into the meaning of angle measurement than, say, a goniometer. Technologically sophisticated measurement tools are familiar to children and include digital stopwatches, thermometers, depth finders, and computer sensors for measuring speed and distance. Choosing the appropriate tool for the attribute to be measured is important.

Formulas can be used to obtain measurements, and whenever possible, mathematics instruction should help students understand these formulas. Many elementary and middle school children have difficulty with understanding perimeter, area, and volume (Kenney and Silver 1997; Lindquist 1989). Often, these children are given formulas such as $P = 2l + 2w$, or $A = l \times w$, or $V = l \times w \times h$ without understanding how that formula relates to the attribute being measured or the unit of measurement being used. Strong connections need to be made between the formula and the actual object. For example, when a student is told the dimensions of a given rectangle—say, 4 centimeters by 8 centimeters—and is asked to find the area using the formula, that student needs to realize that the formula is a shortcut for actually counting the number of square centimeters that it would take to cover the rectangular region.

In another example, students in the upper grades can get a sense for the meaning of the formula for computing the area of a circle ($A = \pi r^2$) with the diagram in figure 3.6, using their knowledge of the area of a parallelogram:

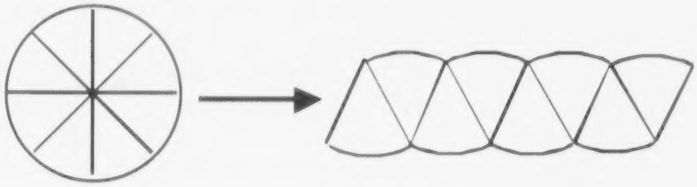

Figure 3.6. Computing the area of a circle.

By grades 3–5, students need to understand the abstraction of a figure on paper that is labeled with measurements such as "10 ft by 35 ft" as a scale representation of an actual rectangle. Scaling, as it is used in maps or in sizing graphing calculator displays, applies to the use of formulas in measurement. A map, for example, is a representation where measurement tools and techniques can be used and where a scale factor relates the distances on the map to the actual distances.

By the secondary school grades, students can handle more sophistication with technological measurement tools and with derived measures. Issues about precision and the accumulation of error become more appropriate at these grades.

Measurement is an area of the pre-K–12 curriculum that lends itself especially well to the use of concrete materials. In fact, it is unlikely that children could ever gain deep understanding of the concepts of measurement without handling materials, making comparisons physically, and measuring with tools. Measurement concepts can grow in sophistication and breadth across the grades, and instructional programs should not repeat the same measurement curriculum each year. Finally, measurement is an important vehicle for highlighting mathematical connections—within mathematics itself and to areas outside of mathematics such as social studies, science, art, physical education, and the student's own interests and experiences.

Standard 5: Data Analysis, Statistics, and Probability

Mathematics instructional programs should include attention to data analysis, statistics, and probability so that all students—

- pose questions and collect, organize, and represent data to answer those questions;
- interpret data using methods of exploratory data analysis;
- develop and evaluate inferences, predictions, and arguments that are based on data;
- understand and apply basic notions of chance and probability.

Elaboration: Pre-K–12

Increasingly powerful and accessible technology has created a significant change in our ability to analyze data. The amount of data available to help make decisions in business, politics, and research is increasing quickly. Consumer surveys are used in the development and marketing of products. Polls are used to determine political campaign strategies. Experiments are used to evaluate the safety and efficacy of new medical treatments. Equally important, statistics are often misused to sway public opinion on issues and to misrepresent the quality and effectiveness of commercial products. A knowledge of statistics is clearly necessary for students to become informed citizens and intelligent consumers, and statistical reasoning needs to be learned. Work in data analysis, statistics, and probability offers a natural way for students to connect mathematics with other school subjects and with experiences they have in their daily lives.

Data are generated from some context, whether collected from observation of all or a sample of existing subjects or produced through simulations. Students should learn to formulate researchable questions; design, carry out, and interpret a survey, study, or experiment; collect relevant data and use it to make a decision; measure how confident they are in their decision; and communicate the results. They can develop conclusions that are accurate and valuable by reflecting on and evaluating the data-gathering process. Possible biases in interpreting data can be monitored, and understanding the process of statistical reasoning can help students develop conclusions that are accurate and valuable.

Answers to questions such as, "How likely is it that this result has occurred because of chance alone?" or "How often would this experiment be likely to yield a particular answer if I did it many, many times?" depend on arguments that are grounded in probability. Young children begin building an understanding of chance and randomness by working with such known populations as the students in their classroom or colored chips in a bag. As they mature, students move from controlled, well-defined situations for which the probability of an event can be readily determined to situations where sampling and simulations help them quantify the likelihood of an uncertain outcome.

A thorough grounding in statistics and probability provides tools and ways of thinking that will be useful throughout students' lives. Because some things children learn in school seem to them predetermined and rule-bound, it is critical that they also learn that some problems involve solutions that depend on assumptions and have some degree of uncertainty. The kind of reasoning used in probability and statistics is not always intuitive, and so it will not be developed in young children if it is not included in the curriculum. Students will benefit from the ability to deal intelligently with variation and uncertainty.

- **Pose questions and collect, organize, and represent data to answer those questions**

In order to understand the field of statistics, students must work directly with data. This means building on the natural curiosity that young children have about their world. Questions that ask, "How many?" "How much?" "What kind?" or "Which of these?" arise quite naturally. Although these questions are not always directly about data, they do offer opportunities with which to begin the study of data. Through activities in which students collect their own data about questions of personal concern, identify attributes of things they encounter every day, and classify objects according to these attributes, students realize that data can be used to investigate phenomena, answer questions, and make predictions.

As students move to later grades, they increasingly generate questions for investigation that are based on current issues and interests. Students in grades 6–8, for example, may be interested in recycling or conservation and may pose the question, "Is it better to use paper or Styrofoam plates in the cafeteria?" Middle grades students are more aware of commercial products and manufacturers' claims, such as "one brand of batteries last longer than another," and they are concerned about issues of fairness. They may ask, "Is our method of scoring fair if we throw out the lowest score and average the others?"

Often questions about issues are not as neatly framed as questions about groups of people or objects. Stating questions in such a way that data can be used to provide an answer can be challenging. The media provide numerous examples of well-formulated questions that yield useful information, as well as examples of questions that are not framed in a way that yields clear results. For example, asking a sample of people which political candidate they prefer may yield some useful information. However, it may be that the persons sampled do not think either candidate is acceptable; they may have been simply indicating which candidate they disliked least. Given these complexities, students in grades 9–12 need to have frequent experiences posing and investigating problems that require using sound principles for designing experiments or surveys.

After posing a useful, well-stated question, students are faced with the task of designing a method for generating data that has the potential to answer the question. Research indicates that time spent in planning for data collection is valuable (Cobb 1998; Roth and Bowen 1994). This does not imply that students need to collect all the data that they use; however, when data collected by others are used, students need to consider the ways in which the data were obtained.

Young children can devise simple data-gathering plans. In elementary grades, the teacher might help frame the question or provide a tally sheet, class roster, or chart on which data can be recorded as they are collected. The "data" might be real objects, such as children's shoes arranged in a bar graph or the children themselves arranged by interest areas. Students learn how to pose questions, structure experiments, and record and code data. In the middle grades, students can learn about bivariate data. By grades 9–12, students should understand the fundamentals of experimental design, potential pitfalls such as bias, confounding variables, and lack of randomness, and the effect these have on the usefulness of any conclusions.

Once data exist, they must be organized and represented in a way that facilitates interpretation. In grades pre-K–2 the idea that data can be ordered is fundamental. First graders can invent ways of ordering their birthday data on a chart so they will know whose birthday comes next (Russell 1991). Students in grades 3–5 can learn more about data representation by analyzing and interpreting representations in books, newspapers, and other media. Teachers can introduce a variety of graphs, tables, lists, and charts. The variety and complexity of representations should grow as students move through the grades. Students in grades 6–8 should begin to compare the effectiveness of representations in organizing the data for further analysis or presenting the data clearly to an audience (Tufte 1983). They consider issues of scale, reflecting on the effect it has on how the data are perceived. As students deal with larger or more complex data sets, technology enables them to reorder data and represent data in graphs quickly so that their focus can move to analysis, understanding, and prediction. Important issues of judgment, such as the choice of the type of representation, the scale used, and labeling, must still be thoughtfully considered by the student.

- **Interpret data using methods of exploratory data analysis**

To use statistical and probabilistic ideas well, students need to see a set of data as a mathematical object. Research has shown that "there is a conceptual leap required in moving from seeing data as an amalgam of individuals, each with its own characteristics, to seeing data as a group with emergent properties, properties that often are not evident in any individual members" (Konold 1998). Just as the young student must develop the abstract idea of a number, such as "5," apart from any particular group of five objects, so must students develop the abstract idea of a set of data. While young children are often most interested in their own piece of data on a graph (*my* birthday is in November, *I* have five people in *my* family), the very act of putting all the students' information in one place begins to draw attention to the nature of the set of data. For example, as first graders make a graph of the number of people in their families, a child may say, "Look, there's nobody at 1," thus sparking a discussion of why 1 is not a possible value in this data set and what the likely range of values might be. Later, although students may still be aware of their individual position in the data, they also begin to describe the set of data as a whole. For example, they may note that "more students come to school by bus than by all the other ways combined" or "after ten days, most of our plants are seven inches tall, but the heights range from three inches to eight inches." By grades 3–5, students are developing ideas about aggregating data and trying to understand what they can say about this new aggregation.

As students begin to see a set of data as a whole, they need to develop competency with the tools available to describe this set. Students need to become increasingly familiar with measures of center or location, measures of spread or dispersion, and attributes of the shape of the data as descriptors of data sets. By working with many sets of data, students in grades 3–8 develop increasing understanding about mode, median, and arithmetic mean. Their understandings are grounded in informal ideas, such as middle, concentration, making things even or fair, or balance point (Mokros and Russell 1995). By the middle grades, computing several measures of center gives students the opportunity to compare the effectiveness of each in representing the entire data set. As an example, consider that either median or mean could be used to express information about average annual rainfall across the state or province; however, if a few pieces of data from communities with very high rainfall skew the data, the median might be the better choice.

The central notion of statistics is making valid statistical comparisons, and this should be a goal across the pre-K–12 grade span. In the elementary grades, students might say that one group has more or less of some attribute than another. By middle school, students should be quantifying these differences by comparing specific statistics. Beginning in grades 4–5 and continuing in middle school, the emphasis can shift from analyzing and describing one set of data to comparisons involving two or more sets (Konold 1998).

At this same grade level, students begin to seriously investigate the relationship between two related attributes or variables. As they move through the middle grades into high school, students add to their statistical repertoire the measures and representations that will help them analyze such relationships. At this time, students will need new tools to identify similarities and differences among multiple data sets, including parallel box plots, back-to-back stem plots, and comparative bar graphs in grades 6–8 and contingency tables, variance, and standard deviation in grades 9–12. Students also need tools to investigate association and trends in bivariate data, including scatterplots and fitted lines in grades 6–8 and residuals, correlation, and regression in grades 9–12.

As data are represented and analyzed, students need to think about the interpretation of their data. What do these data tell them about the question they are investigating? How might they represent the data differently to get better insights? In this part of their analysis, students should become increasingly sophisticated in evaluating the evidence carefully and critically. They should consider how well their conclusions are supported by the data, what alternative conclusions might be drawn, and what factors they might have failed to consider.

- **Develop and evaluate inferences, predictions, and arguments that are based on data**

In early work with data, students often focus only on the actual data they have collected. With a survey of their own class, for instance, students describe and interpret the data while viewing the class as the total population. In the real world, however, most data are gathered from only a sample of the entire population under consideration. Defining an appropriate sample (or samples), collecting data from that sample, describing the sample, and making reasonable inferences relating the sample and the population are central elements of statistical analysis. Upper elementary and early middle school students begin to develop some of the ideas on which statistical inference is based but do not yet have a well-developed understanding of sampling (Schwartz, et al. 1998). Researchers working with students in grades 5–8 have found that students do not expect that "collecting and analyzing data might yield knowledge that is more reliable than their own judgment"(Hancock, Kaput, and Goldsmith 1992). In the later middle grades and high school, students can address the central ideas of sample selection, statistical inference, and quantifying the degree of uncertainty associated with a sample or group of samples. The value of the inferences drawn from a sample can be dramatically affected by many issues, including representativeness of the sample and its size. Students in grades 9–12 can begin to understand these notions.

In addition, students in grades 9–12 should consider factors that might bias a sample. They should also understand that it is possible to quantify the degree of certainty associated with decisions made from data. Forming confidence intervals and reasoning from distributions is not easy. Statisticians and statistics educators who have worked extensively with high school students assert that "notions of statistical inference are subtle and cannot be fully developed until high school"(Scheaffer, Watkins, and Landwehr 1996). Students should leave secondary school with the ability to judge the validity of arguments based on data as they appear in the press.

- **Understand and apply basic notions of chance and probability**

Probability is tied closely to collecting, organizing, and representing data; describing, analyzing, and interpreting data; and inference and prediction. It is also an interesting subject in its own right. It is an area that allows connections to other areas of mathematics, especially number and geometry. Many of the decisions we make in modern life are probabilistic in nature. For example, we need probability "to understand lotteries, insurance, diagnostic medical testing, industrial quality control, weather forecasting, sports handicapping, genetics, and modern physics" (Scheaffer, Watkins, and Landwehr 1996).

In grades pre-K–2 teachers should build on children's developing vocabulary to introduce and highlight probability notions, for example, "We'll *probably* have recess this afternoon." "It's *unlikely* to rain today." In grades 3–5 students can consider ideas of chance in predicting patterns of events. They study these ideas by working with experiments with known theoretical outcomes—using coins, dice, or spinners—and by considering events they know about in the world and categorizing them as impossible, unlikely, likely, or certain, on the basis of their experience.

Many of the phenomena that students encounter, especially in school, have predictable outcomes. If a spinner is one-third red, one-third white, and one-third blue, then a reasonable prediction is that, over time, the spinner will land on blue approximately one-third of the time. Through considering events like these, students come to understand that an individual outcome may be uncertain even when we know something about its likelihood. In a similar example, when one flips a fair coin, it is equally likely to come up heads or tails. Which outcome will result on a given flip is uncertain. What is surprising is that if the event is random and if it is repeated many, many times, then the distribution of outcomes forms a pattern. This emergent pattern runs counter to our intuition. The idea that individual events are not predictable in such a situation but that a pattern of outcomes can be predicted is an important concept that serves as a foundation for the study of inferential statistics.

As students become more sophisticated in their understanding of randomness, they can quantify probabilities in situations with theoretical distributions of outcomes, as with the spinner. If the spinner is one-quarter red and three-quarters white, what is the chance of landing on white? Conditional probability can also be introduced, although research indicates that students may have great difficulty reasoning about probability tasks that involve conditional statements (Shaughnessy 1998). Students also can come to understand how carefully generated samples can reasonably be expected to reflect information about the population from which they are drawn. Sample distributions originally generated through physical simulations should be followed by simulations done electronically using available software, gradually leading to a deeper understanding of these ideas through middle school and high school.

Standard 6: Problem Solving

Mathematics instructional programs should focus on solving problems as part of understanding mathematics so that all students—

- **build new mathematical knowledge through their work with problems;**
- **develop a disposition to formulate, represent, abstract, and generalize in situations within and outside mathematics;**
- **apply a wide variety of strategies to solve problems and adapt the strategies to new situations;**
- **monitor and reflect on their mathematical thinking in solving problems.**

Elaboration: Pre-K–12

The ability to solve problems is not only a purpose for learning mathematics but also a major means of doing so. As students use problem-solving approaches in their investigations of mathematical content, they can develop new mathematical understandings and strengthen their abilities to use the mathematics that they know. Problem solving means engaging in a task for which the solution method is not known in advance. In order to find a solution, students must use their knowledge in different ways and, through this process, may develop new knowledge. Problem solving is an integral part of all mathematics learning, not an isolated part of the mathematics program. It should be a well-integrated part of the curriculum that supports the development of mathematical understanding. Students should have frequent opportunities to formulate, grapple with, and solve complex problems that require a significant amount of effort.

- **Build new knowledge through their work with problems**

How can problem solving help students learn mathematics? Appropriately designed problem situations provide a context within which students can solidify and extend what they know. Well-chosen problems can stimulate deep mathematical investigations.

For example, consider the notion of proportion, one of the major ideas of middle school mathematics. This idea can be introduced through an extended investigation in which students are given different fruit juice recipes (amounts of water and concentrate) and asked which is "fruitier." Since no two recipes yield the same amount of juice, this is a difficult problem for students who do not have an understanding of proportion. As various ideas are tried, students eventually converge on the notion of proportion. This idea emerges as other notions are discarded and as students consider different ways of thinking about proportion and share their ideas with one another.

Problem solving can and should be used to help students develop fluency with specific skills. Waiting until students "know enough" before giving them problem-solving opportunities deprives them of the experience of grappling with a challenge and leaves them bereft of problem-solving experiences. But it is also true that problem solving does not take place in a vacuum; it requires some content knowledge. For example, consider the problem of the teacher who says, "I have pennies, dimes, and nickels in my pocket. If I take three coins out of my pocket, how much money could I have taken?"(adapted from NCTM 1989, p. 24) Knowledge is needed to solve this problem—the knowledge of what pennies, dimes, and nickels are worth and some knowledge of addition. Students do not need to be fluent with addition before working on this problem; in fact, working on this problem provides a good context for practicing addition skills. The important mathematical goal of this problem—helping students to think systematically about possibilities and to organize and record their thinking—need not wait until students can add fluently.

- **Develop a disposition to formulate, represent, abstract, and generalize in situations within and outside mathematics**

People who have developed a mathematical point of view of the world tend to act in ways that are mathematically productive. What behaviors can be expected from people with a mathematical disposition?

Good problem solvers tend to analyze situations carefully in mathematical terms. They are inclined to see if a simple solution is possible before trying something more complicated; yet, they will consider a more sophisticated analysis. For example, a task for middle-grade students presents data about two taxicab companies and ask which company is more reliable (NRC 1998). The quick answer, found by looking at the average time customers had to wait, turns out to be misleading. A more careful mathematical analysis shows that the company whose average time of arrival is shorter also has a much larger variance. In this task, a disposition to analyze more deeply leads to a more complete understanding of the situation and a correct solution.

Individuals with mathematical dispositions tend to explore structure to see what makes things tick mathematically; they abstract and generalize. Such individuals will search for more than one way to approach a problem, thereby making new connections, seeing new components, and revealing different aspects of the mathematics. For example, students in grades 3–5 explore square numbers geometrically, searching for different patterns. One student sees that getting to the next square involves adding an L shape, thereby identifying a formula for the area of the *n*th square in terms of the previous number (see figure 3.3). Another student sees that the area of the *n*th square is the sum of the first *n* odd numbers. These children have learned that there are many ways to look at a problem and that finding one approach is not necessarily a signal that you have finished.

Problem solvers are inclined to explore situations and make conjectures. For example, the following conversation might occur as a group of children in kindergarten class sit in a circle and take turns saying the numbers up to 100:

> Yesterday I said '1'and later on I also said '100.' If I go first, I think I will say the same numbers again today.
>
> No, you won't. Yesterday, Sheila was absent. Today is different.

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Sheila was absent yesterday, but John is absent today. So it will be the same.

Together, the children could explore the situation and make conjectures about what would happen.

Students who are good problem solvers tend to test their conjectures, try to support them through reasoning, and abandon them on the basis of contrary evidence. For example, a group of high school students could attempt to make a pattern for a standard lampshade—one that had a smaller circumference at the top than at the bottom of the shade. They might begin by cutting out a rectangle. They would quickly realize this doesn't work, however, and they should make additional attempts with patterns until finding a solution.

Good problem solvers tend to formulate new problems. For example, a high school class could consider the question of what happens to the vertex of a parabola, $ax^2 + bx + c$, when two coefficients are fixed and the other is changed. (If a and b are fixed and c is changed, for instance, the vertex moves along a vertical line.) A student inclined to formulate a new problem might ask "What would happen if you did the same thing to a cubic, looking at what happens to the relative minimum?"

The teacher plays an important role in the development of problem-solving dispositions by creating an environment in which students are encouraged to explore, take risks, share failures and successes, and question one another. As shown in the examples above, in this type of classroom environment the development of problem-solving dispositions is part of the natural business of learning and doing mathematics.

- **Apply a wide variety of strategies to solve problems and adapt the strategies to new situations**

Problem-solving strategies are part of a student's mathematical tool kit. When a solution to a problem is not readily available, students will benefit from having a repertoire of strategies available to help them make progress. Problem-solving strategies should be treated like any part of the mathematical tool kit. Students should be provided with enough instruction in, and practice with, a range of strategies to enable them to use them. Their use must be embedded in the curriculum so that students develop skill at recognizing when various strategies are appropriate to use and are capable of deciding when and how to use them.

Beginning with the work of Pólya ([1945] 1973), there have been many descriptions of problem-solving strategies. Some of the more frequently cited strategies include using diagrams and other representations, looking for patterns, listing all possibilities, trying special values or cases, working backward, guessing and checking, creating an equivalent problem, and creating a simpler problem. An obvious question is how explicitly should these strategies be taught. As with any other parts of the students' mathematical tool kit, strategies must receive instructional attention if students are expected to learn them (Schoenfeld 1992). In addition, students must become aware of these strategies as they arise and are modeled during classroom activities, and the teacher should encourage

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students to take note of them. For example, after a student has shared a solution and how it was obtained, the teacher may identify the strategy by saying, "It sounds like you made an organized list to find the solution. Did anyone solve the problem a different way?" This verbalization helps develop common language and representations and helps other students understand what the first student was doing.

Some strategies, such as looking systematically for patterns, are very close to content curriculum goals. Looking for patterns is a major aspect of the grades pre-K–2 and 3–5 "Patterns, Functions, and Algebra" section. Other strategies, such as guess and check, are not particularly aligned with specific content. Different strategies are accessible at different ages. This will become more apparent in the grade-band discussions as each attempts to answer the question, "What does problem solving in grades ____ look like?" There are very specific and mathematically powerful strategies, such as proof by contradiction, which can be used when students have achieved a relatively high degree of mathematical sophistication. The reasoning patterns of young children form the precursors to such mathematical strategies. No strategy is learned once and for all, as an algorithm might be learned. Rather, strategies are learned over time and become increasingly complex as they are used in increasingly complex problem situations.

- **Monitor and reflect on their mathematical thinking in solving problems**

There is research (Lester 1985; Schoenfeld 1987) indicating that students' problem-solving failures are often not from a lack of mathematical knowledge, but from ineffective use of what they do know. Effective problem solvers constantly monitor and adjust what they are doing. They make sure they understand the problem. If a problem is written down, they read it carefully; if it is told to them orally, they ask questions until they understand the problem. Effective problem solvers plan frequently. They check what they are doing periodically to see if they seem to be on the right track. If they feel they are not making progress, they stop to consider alternatives and don't hesitate to completely change what they are doing.

For students to become good problem solvers, self-awareness and self-assessment are absolutely essential. Such reflective skills (called "metacognition") are much more likely to develop in a classroom environment that supports them. Teachers play an important role in helping to structure these reflective habits of mind by asking questions such as, "Before we go on, are we sure we understand this?" "What are our options?" "Do we have a plan?" "Are we making progress or should we reconsider what we are doing?" "Why do we think this is true?" Questions like these help students get in the habit of checking their understanding as they go along.

Learning through solving problems is a strong message from the *Curriculum and Evaluation Standards* (NCTM 1989). Classrooms in which students learn to solve problems are those that give students the opportunity to develop the various aspects of problem solving—disposition, strategies, monitoring, and adjustments—in the context of solid mathematics. Teachers in these classrooms encourage an atmosphere of purposeful exploration. Through questioning and dialogue these teachers help their students become aware of the process of problem solving. Problem solving in these classrooms is an important curricular tool as well as a goal in itself.

Standard 7: Reasoning and Proof

Mathematics instructional programs should focus on learning to reason and construct proofs as part of understanding mathematics so that all students—

- **recognize reasoning and proof as essential and powerful parts of mathematics;**
- **make and investigate mathematical conjectures;**
- **develop and evaluate mathematical arguments and proofs;**
- **select and use various types of reasoning and methods of proof as appropriate.**

Elaboration: Pre-K–12

Mathematical reasoning and proof offer powerful ways of developing and codifying insights about a wide range of phenomena. People who reason and think analytically explore the properties and structure of objects and systems. They note patterns or regularities in both real-world and symbolic objects; they ask if those patterns are accidental or if things have to be that way; and they conjecture and prove. Ultimately, a mathematical proof represents the formal codification of patterns of reasoning and justification.

Systematic reasoning of the type described above is a defining feature of mathematics. It is found in all content areas and, with different requirements of rigor, at all grade levels. In the area of number and operation, for example, first graders can note that even and odd counting numbers alternate; third graders can conjecture and justify that the sum of two even numbers will be even; students in grade 6 can determine the likelihood of an even or odd product when two die are rolled and the numbers produced multiplied; and students in grade 10 can be expected to prove, in a variety of ways, that the square of an odd integer is always one more than a multiple of eight. In geometry, elementary students can use manipulatives to determine areas of new shapes. Middle grade students can tear off the angles of a triangle to demonstrate that the sum of the interior angles of a triangle is a straight angle. High school students can prove these properties rigorously. Principled reasoning is at the core of all of mathematics.

Students at all grade levels can engage—in age-level-appropriate ways—in the kind of systematic thinking, conjecturing, and marshaling of evidence that are the precursors to formal mathematical argumentation. Primary grades teachers can make dramatic changes in what they teach once they become aware that the children in their classes are capable of highly sophisticated reasoning (Thompson 1998). By the time students are in secondary school they should be able to approach a problem or mathematical situation systematically, and suggest why what they think is true. Then they should be able to take the next fundamentally important mathematical step and make a compelling argument that shows it must be true. This is the sequence that Mason, Burton, and Stacey (1982) describe as follows: "Convince yourself; convince a friend; convince an enemy."

- **Recognize reasoning and proof as essential and powerful parts of mathematics**

Part of the beauty of mathematics is that when things work, they work for good reason. Mathematics students should understand this. They should expect things to fit together, and they should expect there to be reasons for why things are as they are. Consider, for example, the following "magic trick" one might find in a book of mathematical recreations.

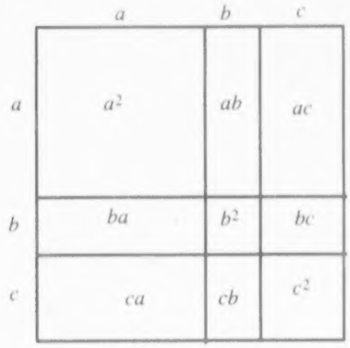
> Write down your age.
>
> Add 5.
>
> Multiply the number you just got by 2.
>
> Add 10 to this number.
>
> Multiply this number by 5.
>
> Tell me the result.
>
> I can tell you your age.

The procedure given to find the answer is, "Drop the final zero from the number you are given and subtract 10. The result is the person's age."

The magical "answer" begs the question, "Why does it work?" This question is a mathematical one. Students at all grade levels can explore and explain problems such as this one. For example, a young student can respond to the question, "I'm thinking of a number. If I double it I get 22. What's my number?" and middle grade students can justify the "magic trick" described above using reasoning and informal algebraic techniques.

In grades pre-K–2, students can develop the precursors of formal reasoning and the understanding that it is important to have reasons for what they say. Questions such as "Why do you think it is true?" and "Does anyone think the answer is different and why do you think so?" can establish the expectation that statements need to be supported or refuted by evidence. Research by Resnick and Omanson (1984) highlights the developmental nature of reasoning about addition. When young children are asked to find "3 + 5," they will typically do so by laying out two sets of three and five objects (possibly their fingers), respectively, and then counting them all: "1, 2, 3, 4, 5, 6, 7, 8." Somewhat older children will recognize that it is not necessary to count the first set, and they will spontaneously start counting from four. At some point, students will—often without instruction—start with the larger number and count from five. This action is based on the recognition that $3 + 5 = 5 + 3$ and that it is more efficient to count starting from the larger number. The teacher who sees this development can raise the issue of what the student is doing and why it is justified. The class discussion that follows can solidify the students understandings; it also can demonstrate that when things work, it is because there is a reason.

Similar kinds of experiences take place at all grade levels. Students in grade 7, for example, can explore properties of a triangle. They notice various relationships such as the congruence of two angles in any isosceles triangle, and they can convince others of the truth of their discovery. Or, they might use properties about triangles to develop and prove conjectures about quadrilaterals, such as "What is the sum of the angles of any quadrilateral?" "Is it always the same?" Students in grades 9–12 can see that the various arithmetic and symbolic manipulations they perform are well justified, as in the case of the area diagram for computing $(a + b + c)^2$, given in figure 3.6 (adapted from Gelfand and Shen 1993, p. 38). The goal is that students make it a matter of habit to ask questions and look for justification.



$$(a + b + c)^2 =$$

$$(a + b + c)(a + b + c) =$$

$$a^2 + ab + ac + ba + b^2 + bc + ca + cb + c^2 =$$

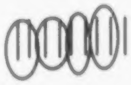
$$a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$$

Figure 3.6. $(a + b + c)^2$.

- **Make and investigate mathematical conjectures**

Some mathematical conjectures have gained notoriety for their simplicity as well as for the challenge they have provided many mathematicians over many years. A famous open problem is known as Goldbach's Conjecture. Goldbach predicted that every even number greater than 4 can be written as the sum of two (not necessarily different) odd primes. For example, "6 = 3 + 3, 8 = 3 + 5, 10 = 3 + 7 = 5 + 5, ..." The conjecture has been tested and found true for numbers into the trillions, but it has not yet been proved or disproved in general.

Such conjectures are so interesting that they often are named after the people who made them and have passed into mathematical folklore. Mathematicians have spent countless hours verifying or refuting them, often motivated by nothing more than curiosity. The point is that doing mathematics involves *discovery*. Conjecture—that is, informed guessing—is a major pathway to discovery. Teachers and researchers agree that students can learn to make, refine, and test conjectures in elementary school. For example, one lesson from a third-grade class includes conversations among students as they looked for patterns when adding even and odd numbers. The students conjectured that the sum of two even numbers will always be even, as will the sum of two odd numbers. They developed representations of even and odd numbers, such as the following representation of the odd number 9:



Representations such as this make it clear how the "extra" unit in 9 can combine with the extra unit in another odd number, yielding an even number (Ball 1989). Similarly, Lampert (1990) analyzes a class session of fifth graders in which the students' task was to figure out the last digit in each of 5^4, 6^4, and 7^4 without doing the multiplication. In considering the same example, Yackel (1998) notes,

> The analysis shows that students moved back and forth between inductive and deductive arguments in the course of the lesson. By the end of the lesson, each student had either made an assertion about a pattern, provided a proof that the pattern would continue, and/or given an interpretation of another student's assertion. This is compelling evidence that elementary students can and do engage in sophisticated mathematical reasoning (p. 8)

At all levels, all branches of mathematics provide opportunities for reasoning and conjecture. Often, a simple shift in the way a task is posed can spark such opportunities. For example, instead of saying, "Show that the mean of a set of data doubles when all the values in the data set are doubled," a teacher can pose the following question: "Suppose all the values of a sample are doubled. What change, if any, is there in the mean of the sample? Why?"

Properly used, technology also opens up the potential for structured exploration. Calculators and computers can now perform with ease operations that were once very costly in terms of time and effort. Dynamic geometry programs, for example, allow students to explore transformations or to examine large numbers of cases where it was previously possible to explore only a few. Graphing programs allow students to explore parameter changes. A variety of computer-based calculus courses now exist in which students explore a wide range of phenomena, such as limits and convergence, before the results are formally introduced.

- **Develop and evaluate mathematical arguments and proofs**

In chapters 4–7, each of the grade-band sections on Reasoning and Proof provides examples that are appropriate for that grade level. In grades pre-K–2, children can demonstrate their reasoning using concrete models. For example, a child might show the teacher by counting out blocks that it does not matter which of two numbers is added first. In grades 3–5, students make mathematical predictions based on observations and begin to provide mathematical justifications for claims they make. For example, a fourth grader might claim that a particular triangle and rectangle have the same area, because each was formed by dividing one of two equivalent rectangles in half. In grades 6–8, students should be able to perceive and explain more complex patterns, as in the case of "figurate numbers" described in chapter 6. In grades 9–12, students can be expected to construct relatively complex chains of reasoning.

Hanna (1998) discusses research about proof, and notes that "Moore states that most postsecondary students have difficulty with formal proof because they 'begin their upper level mathematics courses having written proofs only in high school geometry and having seen no general perspective of proof or methods of proof' (Moore 1994 p. 249)". Asking "why" provides the opportunity for reasoning and proof in many areas and should be a recurring theme throughout the entire curriculum.

An essential component of learning to reason mathematically is learning to evaluate mathematical arguments. It is important that students learn to question assumptions and to make sure that chains of reasoning are justified. Sometimes students make inappropriate generalizations, such as "multiplication makes things larger." Sometimes reasoning mistakes can be subtle, as in the statement, "This number is divisible by 6 and by 4, so it's divisible by 24." Or students may confuse statements with their converses. It is not effective to catalogue the wide range of mistakes students might make and warn them against them, although some errors, such as expanding $(a + b)^2$ to obtain $a^2 + b^2$, occur frequently enough that it is useful to alert students to them. In any case, both plausible and flawed arguments that are offered by the students themselves should create an opportunity for discussion. Classrooms where students are encouraged to present their thinking and in which everyone is encouraged to contribute by evaluating other students' line of thinking provide rich opportunities for developing and evaluating mathematical arguments.

- **Select and use various types of reasoning and methods of proof as appropriate**

There are various kinds of logical arguments. There are traditions for presenting proof in various branches of mathematics, as in, for example, narrative arguments or two-column proofs in geometry or truth tables in logic. Classification plays a major role in various arguments (e.g., "This is an X, so it has the properties of all X's"). There are various forms of algebraic and geometric reasoning, proportional reasoning, probabilistic reasoning, statistical reasoning, and so forth. There are various forms of argument, including proof by analysis of all cases, disproof by counterexample, and proof by application of general results to specific cases. Students need to encounter and build proficiency in all of these at levels of increasing sophistication as they move through the curriculum.

One major goal of the mathematics curriculum is to support the development of reasoning capacity in all students, and the acquisition of tools for constructing proof at appropriate points in their mathematical careers. Students should always be encouraged to think things through carefully, to understand, and to be able to explain. As students' arguments grow more sophisticated, the explanations should increasingly be conveyed in the formal language of mathematics.

Throughout their years in school, students should be provided with opportunities to produce and defend their own chains of reasoning and to examine and critique those produced by others. In the early years, their arguments may be informal and the chains of reasoning short. Later, reasoning and proof are formalized and increasingly complex.

More broadly, it should be stressed that exploring, conjecturing, representing, and proving are all deeply connected aspects of mathematical thinking. "Reasoning" and "Proof" should not be thought of as separable from the bulk of mathematical activity. They cannot simply be taught in a single unit on logic, for example, or by "doing proofs" in a geometry course. Rather, reasoning and proof must be a consistent part of students' mathematical experience in grades pre-K–12. Reasoning mathematically is a "habit of mind," and like all habits, it must be developed through consistent use.

Standard 8: Communication

Mathematics instructional programs should use communication to foster understanding of mathematics so that all students—

- organize and consolidate their mathematical thinking to communicate with others;
- express mathematical ideas coherently and clearly to peers, teachers, and others;
- extend their mathematical knowledge by considering the thinking and strategies of others;
- use the language of mathematics as a precise means of mathematical expression.

Elaboration: Pre-K–12

Communication is an essential part of mathematics education. It is the means by which ideas are shared and a vehicle for clarifying understanding. When students are challenged to think and reason about mathematics and to communicate the results of their thinking to others verbally or in writing, they are faced with the task of stating their ideas clearly and convincingly to an audience. Through experience, students can develop this ability. Moreover, listening to explanations provided by others can afford students the opportunities to develop their own understandings. Conversations in which mathematical ideas are explored from multiple perspectives provide participants with opportunities to sharpen their thinking and to make connections. Such activity also helps students develop a language with which to express mathematical ideas and an appreciation of the need for precision in that language.

Thus, there are dual benefits when mathematics education is rich in communication: students communicate to learn mathematics, and they learn to communicate mathematically. Students who are involved in active discussions in which they justify solutions—especially in the face of disagreement—will gain better mathematical understanding (Cobb and Lampert 1998). Students who have opportunities, encouragement, and support for speaking, writing, reading, and listening in mathematics classrooms will learn to communicate mathematically.

- **Organize and consolidate their mathematical thinking to communicate with others**

Through communication, ideas become objects of reflection, refinement, discussion, and amendment. An important first step in the process is to organize and clarify one's own thinking. Thoughts are ephemeral. Mathematical ideas and arguments are more easily examined and understood when they are articulated in speech or in writing, a process that also helps to build meaning and permanence for the ideas. Students do not retain the mathematics they see and hear in the classroom unless they appropriate it as their own. Reflection is enhanced if students can get the ideas out into the open or commit them to paper. Once the ideas are out in the open, they can be examined, repaired, or augmented.

Reflection and communication are intertwined processes in mathematics learning. Communication for the purposes of reflection can become a natural part of mathematics learning, but this requires explicit attention and planning by teachers. Children in the early grades, for example, can learn to explain their answers and describe their strategies. Teachers need to devote some attention to helping students learn how to talk about mathematics. Then, in order to promote reflection about the discourse, teachers can encourage students to "talk about talking about mathematics" (Cobb, Wood, and Yackel 1994). Thoughtful questions posed by a teacher or classmate can provoke students to reexamine their reasoning.

Writing is also a useful catalyst for reflection. One activity that some teachers find valuable is to use the final few minutes of a class period as a time when students write what they have learned in that day's class and what questions they have about the ideas in the lesson. Such an activity can have value for the students as they reflect on the day's work and clarify their own thinking about the ideas developed in the lesson. This activity also can be useful to the teacher as an assessment. A side benefit is that it also reminds students that they share responsibility with the teacher for the learning that occurs in the lesson (Silver, Kilpatrick, and Schlesinger 1990). Other more elaborate forms of writing that teachers can employ to foster written reflection include journals, learning logs, and portfolio entries. Each of these activities can provide an opportunity for students to organize and consolidate their thinking.

- **Express mathematical ideas coherently and clearly to peers, teachers, and others**

Mathematical ideas gain validity as they are accepted within a community. Within the professional mathematics community, debates periodically occur regarding the acceptability of a proposed proof of a mathematical result. In order for a proof to be recognized as correct, it must gain acceptance within the community. Although students are not trying to produce publishable results of original mathematical research, they benefit from experience with an analogous process. Students need opportunities to test their ideas within the mathematical community of the classroom to see if they can be understood and if they are sufficiently convincing. And, students need to learn what counts as evidence in the mathematics community.

Interaction with others is the primary vehicle through which ideas receive scrutiny and are refined and polished. Those who listen critically and respond with clarifications or corrections are also engaging deeply with mathematics. There is not only a refinement of ideas, but there is also a sense of ownership. Results that emerge in the classroom as one student convinces a skeptical peer of the validity of a mathematical assertion have a vividness and power rarely associated with the justifications one finds on the pages of textbooks. When students work in pairs or in small groups, there are "low stakes" opportunities for students to try out their own tentative ideas and to listen to the reactions and ideas of a small number of classmates. Appropriately structured interactions can help students learn to listen and express their mathematical ideas clearly. These experiences are often useful preparation for presentations in larger forums, such as explaining a problem solution to an entire class. Moreover, when such ideas are worked out in public, all students can profit from being part of the discussion, and the teacher can monitor student learning (Lampert 1990).

In classroom discussions, implicit ideas can be made explicit and then can be analyzed more deeply; everyone can question and learn from them. Students gain insights into their own thinking when they present their methods for solving complex problems or when they justify their reasoning to a classmate or teacher. Through this process, students' own thinking is made explicit, and it becomes possible for them to analyze and improve their ideas. Misconceptions can be identified and corrected, and new learning can take shape.

Teachers need to be skillful in supporting classroom dialogues. First, they must build a sense of community in which students will feel free to express their ideas. For some students, participation in class discussions is a challenge. For example, students in the middle grades are notoriously reticent to stand out in any way during group interactions. Despite this fact, teachers can succeed in creating communication-rich environments in middle grades mathematics classrooms. Support for students is vital, especially for students whose primary language is not English. These students may need some additional support in order to allow them to benefit from communication-rich mathematics classrooms, but they can participate fully if classroom activities are appropriately structured (Borasi, et al. 1998; Fuson et al. 1997; Silver, Smith, and Nelson 1995).

Second, teachers must work to carefully create a climate of appropriate mathematical expectations. In the elementary grades, a major goal is to have students learn to express themselves clearly and coherently so that other students can understand them. By the time students graduate from high school, they should have internalized standards of dialogue and argument so that they habitually present clear and complete arguments. Carefully posed questions and modeling can help to clarify age-appropriate expectations for student work.

Written communication needs to be nurtured in a similar fashion. At all grade levels, students benefit from opportunities to examine and discuss both problematic and exemplary pieces of mathematical writing. By studying exemplary writing, students can develop an understanding of the features of clear mathematical communication. By examining and revising writing that needs improvement, students can incorporate the relevant standards and styles into their own efforts. Since written assessments of students' mathematical knowledge are becoming increasingly typical, students will need practice responding to typical assessment prompts as well.

Young children typically begin school with few writing skills. During their primary years, they may draw pictures, write words, and eventually create sentences. Initially, it will probably be the teacher or parent, rather than the young writer, who makes explicit connections among the various aspects of the student's reasoning. In grades 3–5, students work on sequencing ideas and adding details. Over time, their writing becomes more elaborate. At this level, students also become more explicit about basing their writing on a sense of audience and purpose. For some purposes it will be appropriate to describe their thinking informally via ordinary language and sketches; for other purposes, students will need to communicate in more formal mathematical ways, including the use of appropriate mathematical notation, precise terminology, and carefully drawn diagrams.

Development in written communication continues as students move to the middle grades. In grades 6–8, student writing can be more cohesive and incorporate more mathematical terms and notation, as students extend their capacities for reasoning, analysis, and generalization. This development continues through grades 9–12. The process of learning to write mathematically is similar to that of learning to write in any genre. Practice, with guidance, is important. So is attention to the specifics of mathematical argument, including the use and special meanings of mathematical language and representations and standards of explanation and proof.

- **Extend their mathematical knowledge by considering the thinking and strategies of others**

In the process of working on problems with other students, learners gain several benefits. Often, a student who has one way of seeing a problem can profit from another student's view, which may reveal a different aspect of the problem. One student may conceptualize a problem algebraically while another views it geometrically. Or two students may have different ways of counting the same collection of objects. For example, students who try to solve the following problem algebraically often have difficulty setting up the equations, and they benefit from the insights provided by students who approach the problem using visual representations.

> A farmer has some rabbits and some cages. If he places 2 rabbits in each cage, then he has 4 rabbits left over after all the cages are filled. If he places 4 rabbits in each cage, then he has 2 cages left over after all the rabbits have been placed. How many rabbits and how many cages does the farmer have? (Kruteteskii, 1979)

In their discussions of various approaches to solving this problem, students have opportunities to see the perspectives and methods of others, to evaluate their correctness and utility, and to appropriate them for their own use in later problems.

The process of considering, evaluating, and building on the thinking of others is quite complex. In order to consider various approaches to problems, students need to learn to listen to peers who themselves may still be in the process of developing facility with mathematical concepts and terminology. Students also must learn to question and probe one another's thinking in order to clarify underdeveloped ideas. Moreover, recognizing that not all methods have equal merit, students must learn to examine the methods and ideas of others in order to determine their strengths and limitations. By carefully listening to and thinking about the claims made by others, students learn to become critical thinkers about mathematics. A part of becoming a good mathematical communicator is internalizing such behaviors and learning to review one's own thinking critically. This is more likely to become a lifelong habit if students have frequent opportunities to engage in such behavior in the classroom.

Implicit in this description is the belief that students must seriously engage in learning and assume some responsibility for their own understanding. Also implied is the fact that students need to be working with mathematical tasks about which it is worthwhile to talk. Tasks for which students are expected to have well-developed algorithmic approaches are usually not good candidates for such discourse; however, nonroutine tasks can often be catalysts for rich conversations. Technology is another good catalyst. As students generate and examine numbers or objects on the calculator or computer screen, they have a common (and often easily modifiable) referent for their discussion of mathematical ideas.

- **Use the language of mathematics as a precise means of mathematical expression**

As students articulate their mathematical understanding, they begin by using everyday or familiar language. This provides a base on which to build a connection to formal mathematical language. As stated in the *Curriculum and Evaluation Standards* (NCTM 1989), "Mathematical language builds on the existing structure and logic of common language and connects students' experiences and language to the mathematical world" (p. 80). However, there is sometimes a mismatch between ordinary language and mathematical language. Ordinary language is often imprecise or incorrect when applied to mathematics. For example, words like "area," "factor," and "similar" have special meanings when used in mathematics that differ from their usage in ordinary language. Students need help to build a bridge between their uses of language within and outside the mathematics classroom.

It is important to provide students with experiences that help them appreciate the power and precision of mathematical language. For example, students in grades 6–8 who have difficulty describing various polygonal shapes can appreciate the value of terms like "concave" and "convex" as alternatives to the myriad phrasings in ordinary language that emerge naturally in such discussions, including "pushed in," "like an M," or "normal." Use of precise mathematical definitions should become the norm in middle school and high school. However, it is important to avoid a premature rush to impose formal mathematical language, especially in the elementary grades, but even in high school. Allowing students to grapple with their ideas and develop their own informal means of expressing them can be a powerful way to foster engagement and ownership. Indeed, the confusions resulting from students' imprecision can help them develop an appreciation of the need for precise definitions and for the communicative power of conventional mathematical terms. Teachers need to be sensitive to those times when the benefits of using student-generated language might override the value of mathematical precision.

Technology provides other opportunities and challenges for language development and analysis. The symbols used in a spreadsheet may be related to, but are not necessarily the same as, those generally accepted by mathematicians. Students will profit from experiences that require comparison of standard mathematical expressions and those used in popular applications like spreadsheets or calculators.

Standard 9: Connections

Mathematics instructional programs should emphasize connections to foster understanding of mathematics so that all students—

- **recognize and use connections among different mathematical ideas;**
- **understand how mathematical ideas build on one another to produce a coherent whole;**
- **recognize, use, and learn about mathematics in contexts outside of mathematics.**

Elaboration: Pre-K–12

Connections in mathematics are most evident when students from prekindergarten through high school have the opportunity to see and experience the rich interplay among mathematical topics, within contexts that connect mathematics to other subjects, and in their own interests and experience. Through an instructional emphasis on connections, students not only learn mathematics; they also learn the utility of mathematics and the interrelatedness of mathematical ideas.

Mathematics is not a list of strands or standards, even though it is often partitioned and presented in this manner. Rather, mathematics is an integrated field of study. Viewing mathematics as a whole highlights the need for studying and thinking about the connections within the discipline. With this view, one sees the need for a pre-K–12 curriculum that connects ideas between grade levels, as well as within each grade. This means that teachers must know the needs of their students, as well as the mathematics that the students studied in the grades before and what they will study in the grades that follow. Teachers should build on students' previous experiences and not repeat what students have already done. They should help students to connect their prior knowledge about mathematics as they strengthen their understanding and develop new mathematical topics and ideas each year. This also requires students to be responsible for what they have learned and for using that knowledge to understand and make sense of new ideas.

Viewing mathematics as a connected whole means that ideas from one unit are built upon in subsequent units and that ideas flow within each lesson and from one day's lesson to the next. Cohesive units consist of sets of tasks that connect in such a way as to promote the development of increasingly sophisticated mathematical knowledge. It is within this type of setting that students come to see mathematics as a discipline that is cohesive and to understand that the integration of ideas defines the discipline.

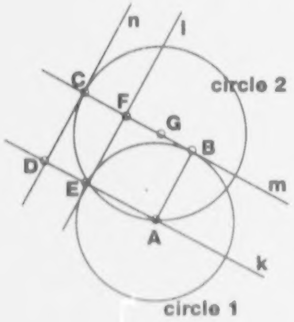
- **Recognize and use connections among different mathematical ideas**

A focus on mathematical connections helps students develop understanding of mathematics and acquire dispositions that are a part of mathematical thinking. For students to develop these dispositions, teachers need to look for opportunities to help students see, use, and communicate about connections. This can be done through the guiding questions that teachers ask during whole-class discussions, conversations with small groups, and work with individual students. A teacher might ask, for example, "How is our work today with similar triangles related to the discussion we had last week on scale drawings?" In many cases, if students are not made aware of mathematical connections explicitly, they will miss opportunities to enhance their development and understanding of mathematics.

In a curriculum focused on mathematics as connected ideas, the notion that mathematical ideas are related pervades students' experiences. From this perspective, new ideas are seen as extensions of previously learned mathematics. Therefore, students should be able to use what they already know to address new situations. Elementary students should be able to link their knowledge of subtraction of whole numbers to the subtraction of decimals or fractions. Middle grade students should be able to recognize and connect multiple representations of the same mathematical idea, such as the ratio that represents rate of change in two variables and the tilt or slope of a line. High school students should be able to connect and extend these representations of slope to solutions of systems of linear equations. Throughout the pre-K–12 span, students should routinely ask themselves, "How is this problem or mathematical topic like things I have studied before?"

- **Understand how mathematical ideas build on one another to produce a coherent whole**

Making connections is a part of what it means to know mathematics and think mathematically. Consider the mathematical thinking that high school students could experience by looking at different relationships that result in the golden ratio (equal to $\frac{1+\sqrt{5}}{2}$ or approximated as 1.61803). First, they could construct the golden rectangle using the directions in figure 3.7 and verify that rectangle ABCD is a golden rectangle.



Instruction for Constructing a Golden Rectangle:

1. Draw two points, *A* and *B*, and construct a segment *AB*.
2. Construct a circle (circle 1) with center *A*, passing through *B*.
3. Construct a line *k*, passing through *A* and perpendicular to the segment *AB*.
4. Let *E* be the point of intersection of circle 1 and line *k*. Construct a line *l*, passing through *E* and parallel to the segment *AB*.
5. Construct a line *M*, passing through *B* and perpendicular to the segment *AB*.
6. Let *F* be the point of intersection of lines *l* and *m*. Construct the midpoint (point *G*) of the segment *BF*.
7. Construct a circle (circle 2) with center *G*, passing through *E*.
8. Let *C* be the point of intersection of line *m* and circle 2. Construct a line *n*, passing through *C* and parallel to the line *l*.
9. Let *D* be the point of intersection of lines *k* and *n*.
10. Rectangle *ABCD* is the golden rectangle.

Figure 3.7. Constructing a Golden Rectangle.

Next, students can find that the ratio of width to length is approximately 1.618. Looking at the Fibonacci numbers, they find the ratio of successive terms in the Fibonacci sequences begins at 1 and approaches the golden ratio (1.61803…). Finally, in another experience, the students find that the solution to the system, $x - y = 1$ and $xy = 1$ also results in an approximation of the golden ratio and its reciprocal. When students question the surprising results that all of these mathematical relationships result in the same ratio, they are beginning to think about the connectedness of mathematics.

As students begin to develop the disposition to think of mathematics as an integrated whole, they should be encouraged to look for connections to help them understand and solve problems. Students should be asking themselves, "Can I look at this another way?" "Is this situation like another one I have seen before?" If students are presented with an algebraic idea, they should think of ways of representing it geometrically, perhaps, if that might deepen their understanding of the idea or help them develop a solution strategy. For example, students should associate a linear equation between two variables with a line in the coordinate plane. This helps explain why a system of two equations in two variables has one solution, no solution, or infinitely many solutions. When deriving the formulas for the surface area and volume of a prism, students should be able to understand and describe the difference between surface area and volume and apply the appropriate procedure (formula) in their calculations. If conceptual understandings are linked to procedures, students will not perceive mathematics as an arbitrary set of rules (NCTM 1989, p. 32). This integration of procedures and concepts should be a basis of school mathematics.

- **Recognize, use, and learn about mathematics in contexts outside of mathematics**

Teachers must help students make legitimate mathematical connections to other areas of study. All curricular content areas have mathematical components, and teachers should emphasize the mathematics inherent in the study of these different subjects. However, when doing so, teachers need to recognize and make explicit the mathematical characteristics of the activity. In attempting to make connections, teachers should also be sure that the mathematics content is not oversimplified. The purpose of curricular connections is to deepen students' understanding and increase students' engagement in mathematics.

The opportunity for students to experience mathematics in a context is important. The context can be generated from within mathematics or from a context arising in other disciplines. Of all disciplines, science probably has the most obvious connections with mathematics. The link between mathematics and science is not only through content, but also through process. The processes and content of science can inspire an approach to solving problems that applies to the study of mathematics. In the *National Science Education Standards*, a yearlong science activity for elementary school children is described (NRC 1996, pp. 131-133). In this activity, students design instruments for measuring weather, make measurements, and organize and communicate their data. The connections to mathematics are substantial.

Using mathematics in applied situations leads to deeper understanding. In another example, the relationship between the diameter and circumference of a circle can be studied empirically by collecting a variety of circular objects and measuring their circumference and diameter. Students in grade 6 can conduct an experiment to see if there is a relationship between the two variables, circumference (C) and diameter (d). By collecting data and graphing these data, students can see that the points all lie close to a straight line through (0,0). This suggests that the ratio of C/d is constant. Students usually get an average value for C/d between 3.1 and 3.2. Students should realize that this does not prove that C/d is a constant or show that its value is 3.14..., but it does give suggest a conjecture that C/d is a constant.

Finally, consider the use of data analysis and statistics to help students clarify issues related to their personal lives. Students in grades pre-K–2 who are working on calendar activities can collect data on the weather by recording rainy, cloudy, or sunny days. They can record the data, count days, generalize about conditions, and make predictions for the future. Students in grades 3–5 can use the Internet to collaborate with other classrooms to collect and analyze data as they recreate Eratosthenes' method for calculating the circumference of the earth (Stevens 1998).

Students in grades 6–8 are often intrigued with their classmates' attitudes, interests, and social relationships. Using survey data, students can link data analysis and statistics to learn more about social studies. The example of "selecting delegates" (Lappan et al. 1996) shows the connections to mathematics. In this example, students determine the number of delegates to attend an environmental conference from across the United States. The criteria for delegate selection—geographic regions represented, gender, and ethnicity—assist in ensuring that a fair decision is made. This use of ratio, proportion, and percents extends students' understanding of mathematics. Students set up a ratio of the population of the geographic regions to the overall population of the United States and find the equivalent ratio for delegates from each geographic region to the total number of delegates. A table with data about the Unites States population helps answer specific questions about the information, such as "How many people live in the New England region?" Mathematics helps answer the important social questions, including "How many of the delegates should be chosen from each of the regions?"

Students in grades 9–12 can use data analysis to help them become informed and responsible citizens. They can collect and analyze data related to current events, government appropriations, and elections, for example. These activities help students understand how statistics and data analysis can play an important role in decision-making in civic arenas such as county and provincial government and social service.

Standard 10: Representation

Mathematics instructional programs should emphasize mathematical representations to foster understanding of mathematics so that all students—

- **create and use representations to organize, record, and communicate mathematical ideas;**
- **develop a repertoire of mathematical representations that can be used purposefully, flexibly, and appropriately;**
- **use representations to model and interpret physical, social, and mathematical phenomena.**

Elaboration: Pre-K–12

The ways in which mathematical ideas are represented has a fundamentally important role in shaping the ways people can understand and use those ideas. To make the point simply, consider the following arithmetic questions:

> What is the product of the two numbers represented by the Roman numerals MCMLXXXVII and MDCCXLIII? What is the quotient when the first is divided by the second?

In the given representation, multiplication is exceedingly difficult and division is seemingly impossible. Yet, when the representations are converted to standard base-ten notation, it a straightforward process to find the product and quotient, respectively, of the numbers 1987 and 1743.

This example illustrates a general point that mathematical symbolism and representation is one of the most significant achievements of humankind. Mathematical representations have been polished and refined over centuries. When students gain access to mathematical representations and the ideas that are represented, they have a set of tools that provides a significant expansion of their capacity to think mathematically.

Representation is a multifaceted concept, and there are subtle and complex issues of language associated with it. The term "representation" refers both to process and to product—in other words, to the act of capturing a mathematical concept or relationship in some form and to the form itself. Moreover, the term applies both to externally observable processes and products and to those that occur "internally" in the minds of people doing mathematics. All these meanings of representation are important to consider in school mathematics.

Some forms of representation—such as diagrams, graphical displays and symbolic expressions—have long been part of school mathematics. Unfortunately, these representations and others have often been taught and learned as if they were ends in themselves. This approach limits the power and utility of representations as tools for learning and doing mathematics. Representations should be treated as critical elements in supporting student understanding of mathematical concepts and relationships; in communicating mathematical approaches, arguments, and understanding to one's self and to others; in recognizing connections among related mathematical concepts; and in applying mathematics to realistic problem situations via modeling. In addition to these long-standing reasons for including representation as an important process of school mathematics, new forms of representation associated with electronic technology create a need for even greater instructional attention.

- **Create and use representations to organize, record, and communicate mathematical ideas**

As the example of Roman numerals versus Arabic base-ten representation indicates, the forms of representation that are used will shape one's capacity to understand and do mathematics. The fact that it took centuries to develop the base-ten representation highlights another point. That is, many of the representations we now take for granted—such as numbers expressed in base-ten or binary form, fractions, equations, graphs, and spreadsheet displays—are the result of a process of cultural refinement that took place over many years. These tools are extremely powerful for solving problems. Equally important, a shared mathematical language is fundamentally important for purposes of communication. Written representations are central to reading and understanding mathematical text written by others. They also can be very useful in sharing with others one's own understandings about and approaches to mathematics. For these reasons, it is important that students learn conventional forms of representation in ways that facilitate their learning of mathematics and their communication with others about mathematical ideas.

Mathematical representations do not exist in isolation. They are usually developed and applied to various classes of situations and are connected to particular bodies of mathematical concepts. Part of understanding a representation involves having links to related mathematics, knowing the circumstances to which it applies, and knowing how that representation can be used in the service of particular goals. Given a particular problem, one might choose a representation that would help clarify and solve that problem. A different problem dealing with the same phenomenon might be more profitably approached with a different representation. Sometimes the treatment of conventional representational forms in school leads students to think of them as arbitrary and disconnected both from settings of use and from students' other knowledge and experience. Conventional representations should be viewed as powerful tools for organizing and communicating mathematical thinking about mathematics. It is useful to consider not only how conventional representations might be better taught to students, but also to consider the role that other, less conventional, representations might play in the mathematics classroom.

The fact that representations are such effective tools may obscure how difficult it was to develop them and, more importantly how much work it takes to understand them. Teachers in the elementary grades are aware of how difficult base-ten notation is for young children, and the curriculum allows room for a large amount of give-and-take between students' emerging understanding of the counting numbers and the structure of base-ten representation. But as students move through the curriculum, the focus is increasingly on presenting the mathematics itself, perhaps under the assumption that students who are old enough to think in formal terms do not, like their younger counterparts, need to negotiate between their naive conceptions and the mathematical formalisms. Research indicates, however, that students at all levels need to work at developing their understandings of the complex ideas captured in conventional representations (Greeno and Hall 1987). A representation as seemingly clear as the variable "x" can be difficult for students to comprehend.

The idiosyncratic representations constructed by students as they solve problems and investigate mathematical ideas can play an important role in supporting the development of students' understanding in mathematical learning and performance. Although personally constructed representations often lack the precision and generality of conventional representations, they appear to be valuable to students. In particular, they support the understanding and solution of problems; they provide meaningful ways to record a solution method and to describe the method to others; and they provide an experiential base from which students can develop an appreciation of the nature and power of other representations, including conventional ones. Representations constructed by students also give teachers opportunities to gain valuable insights into students' ways of interpreting and thinking about mathematics. Using this information, teachers can build bridges from students' personal representations to more conventional ones, when appropriate. In summary, it is important that students have opportunities not only to learn conventional forms of representation, but also to construct, refine, and use their own representations as tools to support learning and doing mathematics.

Computers and calculators change what students can do with conventional representations and expand the set of representations with which students can work. For example, students can flip, invert, stretch, and zoom in on graphs using graphing utilities or dynamic geometry software; they can use computer algebra software to manipulate expressions; and they can investigate complex data sets using spreadsheets. As students learn to use these new, versatile tools to enhance their methods for organizing and recording their ideas, increase their understanding of mathematical concepts and relationships, and facilitate communication of their thinking, they also can consider ways in which some representations used in electronic technology differ from conventional representations. For example, scientific calculators use a representational form for a number in scientific notation that differs from the form typically found in a textbook. A symbolic manipulator often changes the order or form of an algebraic expression. These technological tools offer students and teachers many new issues related to mathematical representation.

- **Develop a repertoire of mathematical representations that can be used purposefully, flexibly, and appropriately**

There are complex relationships among mathematical objects and the forms used to represent them. Often the same object or relationship can be conceptualized or represented in different ways, and those different representations might be appropriate for different purposes. At various times, for example, one might want to think about "48" in different ways, such as "4 tens and 8 ones," "3 tens and 18 ones," "2 less than 50," "the area of a 3-by-16 rectangle," "the perimeter of a square with a side of 12," "a little less than half of 100," and "a little less than 7^2." Each of these representations of 48 is likely to be very useful for some purposes and less useful for others. It is important for students to be able to move flexibly among such representations, choosing or creating representational versions that are useful for a particular purpose.

Different representations often illuminate different aspects of a complex concept or relationship. For example, students usually learn to represent fractions as sectors of a circle or as pieces of a rectangle or some other figure, sometimes using physical displays of pattern blocks or fraction bars. This form of representation is useful in conveying the part-whole interpretation of fractions. It supports the development of an understanding of fraction equivalence and it can help students visualize the meaning of the addition of fractions, especially when the fractions have the same denominator and when their sum is less than one. Yet this form of representation does not convey other interpretations of fraction, such as ratio, indicated division, or fraction as number. Nor does it support directly an understanding of the addition of fractions whose denominators are different or whose sum is greater than one. Similarly, other common representations for fractions, such as points on a number line or ratios of discrete elements in a set, convey some but not all aspects of the complex fraction concept. Thus, in order to become deeply knowledgeable about fractions—and many other concepts in school mathematics—students will need a variety of representations that support their understanding.

One of the powerful aspects of mathematics is its use of *abstraction*—the fact that symbolization strips away some features of a problem that are not necessary for analysis and that the "naked symbols" can be operated on easily. Moreover, recognizing that a particular object has a particular representation may provide a large amount of information about the object itself. In many ways, this fact lies behind the power of mathematical applications and modeling. For example, a problem may be stated in terms of the real world context that generated it:

> From a ship on the sea at night, the captain can see three lighthouses and can measure the angles between them. If the captain knows the position of the lighthouses from a map, can the captain determine the position of the ship?

When one translates the problem into a mathematical representation, the ship and the lighthouses become points in the plane, and the problem can be solved without knowing that the problem is about ships. Many other problems from different contexts may have similar representations. As soon as the problem is represented in some form, the classic solution methods for that mathematical form may solve the problem. For instance, in mathematical modeling problems, if the phenomenon being modeled turns out to be a quadratic function, students should know that there is one critical value that can be determined analytically, that the graph is symmetric, and so forth.

Common underlying representations can also be useful in helping students build connections as they learn mathematics. Students who can represent the product $(a + b)(c + d)$ as a rectangle whose area is composed of smaller rectangles with areas of ac, ad, bc, and bd have a way to link their study of the multiplication of algebraic expressions (e.g., $(x + h)^2 = x^2 + 2xh + h^2$) to their previous study of multiplication of two-digit numbers (e.g., $15 \times 27 = (10 \times 20) + (10 \times 7) + (5 \times 20) + (5 \times 7)$). In this way, representations enable students to make connections across topics in the mathematics curriculum that would otherwise seem disconnected.

Technological tools now offer opportunities for students to have more and different experiences with the use of multiple representations. For example, several software packages allow students to view different representations of a function simultaneously. For example, a function can be displayed in tabular, graphical, and equation form. Such software can allow students to examine how certain changes in one representation, such as varying a parameter in the equation $ax^2 + bx + c = 0$, simultaneously affect the other representations. Similarly, dynamic geometry software allows students to consider geometrical, computational, and symbolic representations simultaneously.

During grades pre-K–12 students should encounter multiple representations for many mathematical concepts and relationships. As they expand their representational repertoire, it is also important for students to reflect on their use of these representations to develop an understanding of the relative strengths and weaknesses of various representations for different purposes. For example, students learn different representational forms for displaying statistical data, and it is important that their learning include opportunities to consider the kinds of data and questions for which it might be more appropriate to use a circle graph rather than a line graph, or a box-and-whiskers plot instead of a histogram.

- **Use representations to model and interpret physical, social, and mathematical phenomena**

The term "model" has many different meanings. So, it is not surprising that the word is used in many different ways in discussions about mathematics education. For example, "model" is used to refer to physical materials with which students work in school, as in manipulative models. The term is also used in ways that suggest exemplification or simulation, as in when a teacher "models" the problem-solving process for her students. Yet another usage treats the term as if it were roughly synonymous with representation. Amidst this array of uses and meanings is another that is particular to mathematics, and it is the major focus of this part of the discussion about the role of representation in school mathematics. The term "mathematical model" in this context means a mathematical representation of the elements and relationships within an idealized version of a complex phenomenon. Mathematical models can be used to clarify understandings of the phenomenon and to solve problems. What it means "to model" in this sense includes not only representation, but also acting upon the representation and interpreting the meanings of one's actions within the mathematical model and with respect to the phenomenon being modeled.

Many of the symbolization-and-analysis activities in which students engage are a simplified form of modeling in which situations are characterized symbolically and then information about the situation is derived analytically. Consider, for example, a typical optimization problem in which students are asked to determine which price for a magazine will yield the highest profits, given that each time the price of the magazine is increased, the number of people who will buy it decreases. To solve such a problem, a student might determine an equation expressing profit as a function of magazine cost and then use graphical or analytic techniques to determine the best price. Although classroom versions of such tasks are often dramatically simplified, the work captures some aspects of the modeling process.

There are activities in which models allow a view of a real-world phenomenon through an analytic structure imposed on it. One example is building a model of traffic flow. An example of a general question explored might be, "How long should a traffic light stay green to let a reasonable number of cars flow through the intersection?" Students can gather data about how long (on average) it takes the first car to go through, the second car, and so on. They can represent these data statistically, or they can construct analytic functions to work on the problem in the abstract, considering the wait time before a car starts moving, how long it takes a car to get up to regular traffic speed, and so on. This kind of analysis allows for a simulation of traffic flow, which can be used to make judgments about how long a light should be left green.

Technological tools now allow students to explore iterative models for situations that were once studied in much more advanced courses. For example, it is now possible for students in grades 9–12 to model predator-prey relations. The initial set-up might be that a particular habitat houses so many wolves and so many rabbits, which are the wolves' primary food source. When the wolves are well fed, they reproduce well (and more wolves eat more rabbits); when the wolves are starved, they die off. The rabbits multiply readily when wolves are scarce, but lose numbers rapidly when the wolf population is large. Modeling software that uses difference equations allows students to input initial conditions and the rules for change and then see what happens to the system dynamically.

The act of modeling is a complex enterprise. One oversimplified diagram representing the process is given in figure 3.8.

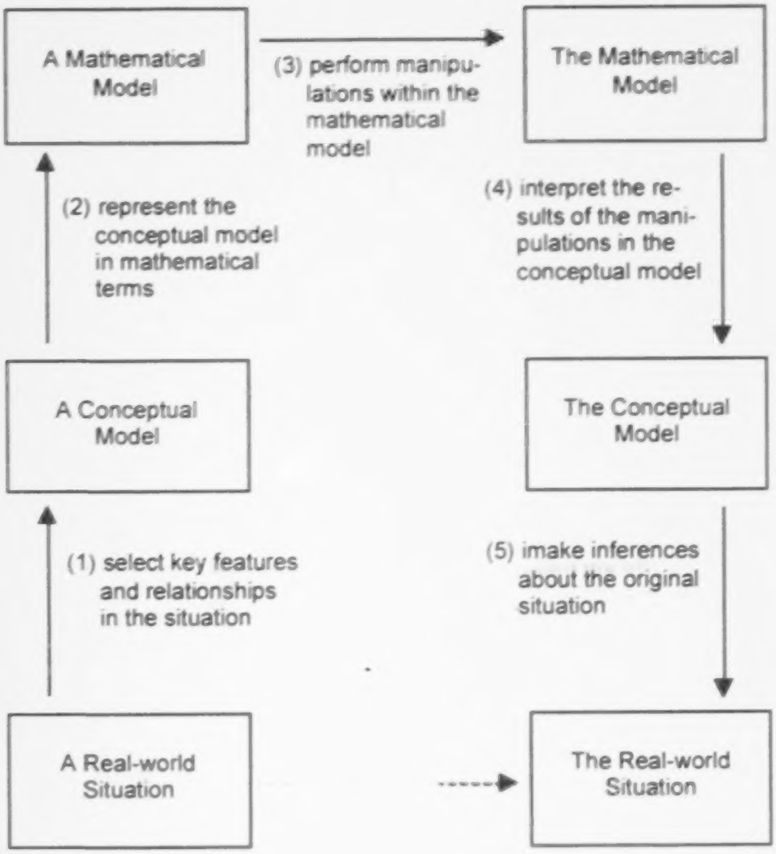

A Mathematical Model
(3) perform manipulations within the mathematical model
The Mathematical Model
(2) represent the conceptual model in mathematical terms
(4) interpret the results of the manipulations in the conceptual model
A Conceptual Model
The Conceptual Model
(1) select key features and relationships in the situation
(5) imake inferences about the original situation
A Real-world Situation
The Real-world Situation

Figure 3.8. Aspects of the modeling process.

A "tour" of figure 3.8 from the bottom left up to the top left (representing the modeling process), across the top (representing mathematical analyses) and down to the bottom right (representing the interpretation of the findings) suggests the path by which conclusions are drawn about the situation being modeled. Each of the three examples described above works in the way suggested here. Note, however, that each stage of the process is potentially problematic. Are the right features selected for analysis? Does the mathematical model capture the right relationships among them? Are the manipulations within the model justified and appropriate? Is the interpretation meaningful? And does it make sense when mapped back to the original situation? All of these questions must be kept in mind when students are engaged in the modeling process.

In summary, representations are vehicles for mathematical thought. Individuals often build their own idiosyncratic representations as a way of understanding ideas and relations. Society offers the product of its collective wisdom in conventional mathematical representations, and these have great power for problem solving and communication. One of our major tasks in classrooms is to bring these two together and provide students meaningful access to those powerful societal tools.

Conclusion

Chapters 4–7 of this draft discuss in detail the ideas that have been introduced in this chapter. Although ten standards have been chosen in an effort to show how mathematical ideas as well as students' learning should progress over the pre-K–12 span, these overviews are, at best, a sketch of the goals that teachers and students might hope to achieve. In the grade band chapters that follow, a more detailed image is conveyed.

Because the standards have been stated somewhat generally in order to extend across the grade-bands, and because there are four grade-band divisions, a considerable amount of detail and guidance for teachers is embedded in the bulleted subtopics that elaborate each of the content standards' focus areas. For the content standards, the image is conveyed largely through description of what students should know and be able to do with the focus areas. For the process standards, elaboration is provided through questions. It is, of course, not possible to describe fully what would occur for all students at all grade levels, but the intention of the grade-band discussions is that teachers and others will be able to use these ideas as they collaboratively and deliberately structure their own instructional programs, within all of the constraints and contextual features they must address.

#4

Reader Reaction

Examples, tasks, problems, and instances of student dialogue are used throughout this draft.

- **Which ones do you find most helpful? Can you suggest other that will help make certain points?**
- **Would you like to see more examples? Of what type?**

#5

Reader Reaction

The following four chapters discuss the standards for each grade band.

- **Are the messages consistent across the grade bands? Where are there inconsistencies or contradictions?**
- **How do you like the tone and style of these chapters?**

Chapter 4
Standards for Grades Pre-K–2

Mathematics Education before Kindergarten

Between infancy and kindergarten the foundation for children's mathematical development is established. Mathematics learning should build on children's curiosity and enthusiasm and should grow naturally from their experiences. If done well, early mathematics is more than "getting ready" for school. It is not about accelerating young children into elementary arithmetic. Rather, mathematical experiences for children, if appropriately connected to children's worlds, challenge young children to explore ideas in more sophisticated and rich ways than previously believed possible. These ideas include number as well as quantitative relationships, shape, space, symmetry, and patterns.

Many children are cared for by family members during their entire preschool years. Other infants, toddlers, and preschool children receive some type of care and education on a regular basis from persons outside their families. Increasingly, mathematics education will be offered to more preschoolers in more formal settings, as several states are instituting universal preschool education. Various federal and state agencies provide financial support for preschool programs designed to facilitate academic achievement. Such change is prompted by the understanding that children's innate desire for learning must be nurtured in appropriate, supportive ways so that all children begin formal schooling on a "level playing field." All preschool children need opportunities to explore their world and experience mathematics through their play.

The ideas addressed in this section speak to experiences that prepare children for mathematics instruction in school. Following this section are the ten content and process standards that elaborate experiences for school mathematics during the pre-K-2 years. For purposes of discussion, the term preschool is used to refer to all children who are not yet enrolled in kindergarten.

Why do we need to focus on mathematics before kindergarten?

The principle that all children can learn mathematics applies to children of all ages. Many mathematics concepts, at least in their intuitive beginnings, develop before school. Infants spontaneously use the ability to recognize and discriminate small numbers of objects (Starkey and Cooper 1980). Before they enter school, many children develop knowledge about number and geometry. Young children possess a substantive informal knowledge of mathematics. They use mathematical ideas in everyday life and develop informal mathematical knowledge that can be quite complex and sophisticated (Clements, Swaminathan, Hannibal and Sarama in press; Ginsburg, Choi, Lopez, Netley, and Chi 1997). The long-term success of children's learning and development requires quality experience during their "early years of promise" (Carnegie Corporation, 1998). Quality experience for children is most likely to occur in environments that are rich in language, encourage children's thinking, value children's uniqueness, and nurture children's explorations. Children who experience high-quality early childhood education have a better chance of achieving to higher levels in school than those who do not have that experience.

What types of mathematical experiences are most appropriate before kindergarten?

Because young children develop a disposition for mathematics based upon their early experiences, opportunities for learning need to be positive and supportive so that children learn to trust their own abilities to make sense of mathematics. Parents, teachers, and other caregivers can help children become aware of the mathematics in their activities through sensitive observation, conversation, and guidance. Early mathematical experiences include situations in which a parent places crackers in a toddlers' hands while saying, "Here are two crackers—one, two", or when a three-year-old chooses how she wants her sandwich cut—into triangles, rectangles, or small squares. A child who has taken a long jump and wants to show her grandfather may be given a piece of string to use. As a child arranges stuffed animals by size, an adult might ask, "Which animal is the smallest?" or "Which is the largest?" When children recognize a stop sign from the back, focusing on the octagonal shape rather than the red background and the word "STOP," adults have an opportunity to talk about different shapes in the environment. Caregivers in any setting can help children look for mathematical connections and relationships, encourage their questions, and promote mathematical discussion about topics that interest children.

Mathematics is developing as playmates create streets and buildings in the sand or make playhouses with empty boxes. It grows as children count steps across the room or sort collections of rocks and other "treasures." Children often compile a wealth of knowledge about topics that interest them, so mathematics should be developed from children's interests and activities. Through their everyday activities children learn mathematical concepts: sorting (putting toys or groceries away), spatial relations and comparisons of solids (building with blocks), representation (drawing to record their ideas), patterns (going about daily routines), directional terms (singing motion songs), and spatial visualization (working puzzles). Young children represent numbers and shapes with objects and with mathematical actions, such as counting or transforming shapes. They show what they know by talking, as well as through models, role playing, and pictures. In the preschool years, quality learning is often incidental and informal. This does not mean unplanned or unsystematic. The most powerful mathematics learning for a three-year-old is seldom acquired sitting down in a group lesson.

Adults have opportunities to monitor and influence how children spend their time. For example, if children are allowed to watch television, an hour spent watching *Sesame Street* might be a better choice for helping them learn mathematical ideas than other viewing options. Activities that encourage mathematical learning—such as books and stories with numbers and patterns or music with actions and directions like up, down, in, and out or games that involve rules and taking turns—help children understand a range of mathematical ideas. Things to count, sort, compare, match, put together, and take apart should be accessible to children.

It is especially important to provide introductions to the language and conventions of mathematics, always starting with and maintaining a connection to children's informal knowledge and language. Relevant mathematics language should be infused into meaningful contexts. Young children are grappling with number words and counting for the first time. They need to learn words used to compare and to indicate position and direction. For example, a friend may be *taller* than they are, but the same friend may be *shorter* than the girl next door. Children gain both competence and ownership as they return to favorite activities repeatedly (such as songs and finger plays that count forward and backward).

If children have rich mathematical experiences during the preschool years, the typically wide variation in children's knowledge upon entering school can be narrowed. "Not knowing" is a reflection of not having had opportunity to learn rather than inability to learn. At-risk children who need additional support so that they do not start school at a disadvantage should be identified with appropriate assessments. Children showing early signs of developmental or learning problems require immediate attention. Pediatricians and other health care providers can recognize indicators of early learning difficulties and can suggest resources to accommodate these challenges.

Assessments used for screening children who are about to enter kindergarten must be adapted to the needs and characteristics of young children. They should include a variety of mathematical emphases beyond number. Such early assessments should be used to gain information for teaching and for potential early interventions, rather than for sorting children. Interviews and observations are more appropriate assessment strategies than conventional group tests, which are not likely to yield reliable, valid, or complete data.

Mathematics Education Pre-K–2

The pre-K–2 span is one of profound developmental change. In no other grade band is the growth in mathematical knowledge so remarkable. In this chapter, pre-K–2 children are considered as a group, although it is important to recognize that children develop mathematical ideas at different times and in different ways, that many different educational settings serve young children, and that the age at which children begin formal educational programs varies. Those who teach young children—parents, caregivers, or school teachers—need to be aware of the earliest beginnings of mathematical ideas as well as expectations for school mathematics during the primary years.

Developing a solid mathematical foundation in the early primary grades means nurturing children's confidence that mathematics learning is within their reach and facilitating their success. High quality early childhood mathematics programs:

- Build upon and extend children's intuitive and informal mathematics knowledge
- Are grounded in knowledge of child development
- Provide environments that encourage children to be active learners who are eager for new challenges
- Develop a strong conceptual framework that provides anchoring for skills acquisition
- Nurture and develop children's natural inclination to solve problems.

During the pre-K–2 years, children are building beliefs about what mathematics is, what it means to know and do mathematics, and about themselves as mathematics learners. These beliefs influence their thinking, performance, attitudes and decisions related to studying mathematics in later years. Therefore, it is imperative to provide all students with programs that include significant mathematics in a manner that honors both the mathematics and the nature of pre-K–2 children.

A Climate for Young Learners

Most children enter school confident with their own abilities and curious and eager to learn about numbers and other mathematical objects. In general, young children devise their own methods to find solutions to mathematical problems. Teaching in the early grades should encourage and build on the strategies of children to develop more general ideas and systematic approaches in mathematics. Success at later grade levels depends largely on the quality of the conceptual foundation established during the first years of school.

Young children are active, resourceful individuals who construct, modify, and integrate ideas by interacting with the physical world and with other children and adults. They learn by talking about what they are thinking and doing and by collaborating and sharing their ideas. Their ability to communicate through language, pictures, and other symbolic means develops rapidly during these years. Young children also make connections that clarify and extend their knowledge; they add new meaning to past experiences by writing and interpreting others' written records. Adults facilitate children's development of mathematics by asking questions that lead to clarifications, extensions, and the development of new understandings. Language is very important to learning mathematics.

Young children make sense of the world by reasoning and problem solving, and teachers should recognize that young children think in ways that can be sophisticated. At the same time, it should be recognized that a child's ways of knowing and communicating is different from that of an adult. Children will find their own ways of representing and communicating their ideas. By the end of the second grade, children should begin to use many conventional representations with understanding.

As noted earlier, early experiences establish a base for further mathematical knowledge that grows in school settings. Classrooms should include materials to count, sort, measure and represent numbers and patterns. Children continue to learn through their constructions with interconnecting blocks and clay, games and puzzles, stories and dramatic play, music and art.

Mathematical concepts develop throughout the pre-K–2 years and at different times and rates for each child. All children need adequate amounts of time and opportunities to develop, construct, test, and reflect upon their understanding of mathematics. Since a wide range of cognitive development is related to previous experiences and opportunities to learn, primary education must build on the idea that all children can learn significant mathematics, and the school must take the responsibility for supporting that learning. That is, strong mathematics programs provide the support and resources so that all children leave grade two confident and competent in the mathematics described for this grade band.

Mathematics for Young Learners

Each of the standards presented in this draft document has formal beginnings in the pre-K–2 years. The ten standards, rather than being separate topics for study, are designed to support the development of important, connected mathematical ideas and should be closely interwoven. At the core of mathematics in grades pre-K–2 are the number and geometry standards. Each of the other mathematical standards, including patterns, measurement, and data and the process standards, contribute to and are learned through the number and geometry strands.

Children's number development moves through a pattern of increasingly sophisticated levels of constructing ideas and skills. In particular, children learn to count, group, combine, and separate numbers, sets, and shapes as they build their concepts of place value and the four arithmetic operations. The big ideas in number include the many meanings of numbers; systems of counting numbers (i.e., one-to-one correspondence, increasing and decreasing by one more and less, skip counting); composing and decomposing numbers, place value relationships, and part-whole relationships. Children develop understanding of space and geometry through their own movement and based on visual and tactile explorations. Two- and three-dimensional shapes, properties of shapes (including symmetry) and transformations are encountered and explored in the early primary grades.

The discussion of the ten standards in this chapter reflects the inter-relatedness of content and process in young children's learning of mathematics. Children recognize patterns of all sorts in the world around them and gradually begin to use patterns as a strategy for solving problems. Both number and geometry are used in measurement, for example, as children answer questions such as "How heavy?" "How big?" "How long?" and "How tall?" The ability to gather, organize, represent, and use data to answer questions is likely to involve all of the process standards. Skills are acquired in ways that make sense to children. Teachers must continually attend to maintaining a balance between emphasis on conceptual and procedural (skill) aspects of mathematics. Children whose skills and strategies are based on understanding of fundamental mathematical concepts are more likely to retain and be able to expand their knowledge and understanding in later years.

The pre-K–2 mathematics program should take advantage of technology. Guided work with calculators can enable children to explore number and pattern, focus on problem-solving processes, and investigate realistic applications. Through their experiences, and with teacher support, children come to recognize when using a calculator is appropriate and when it is more efficient to compute mentally. Computer environments also can make a powerful and unique contribution to the mathematics program. They enable children to manipulate virtual representations of numbers and see, for example, that "4 tens and 7 ones" is the same as "3 tens and 17 ones." Simple programming environments such as Logo can be used to engage students in geometric and spatial investigations.

Children frequently possess greater knowledge than they are able to express in standard symbolic notation, and teachers need to determine through assessment what children already know and what they still have to learn. Information from a wide variety of classroom experiences—classroom routines, conversations, written work (including pictures), and observations—provides the teacher with an assessment of what young children already know, are in the process of learning, and still need to know. Quality assessments inform instructional decisions and allow teachers to monitor individual children's progress while focusing on how children are thinking about mathematics.

Mathematics learning for pre-K–2 children is active, rich in mathematical language, and filled with thought-provoking challenges. Educational research and classroom experience together challenge us to hold high expectations for all children and look at what they can do *with understanding* rather than to plan programs around preconceptions about children's limitations. This does not mean abandoning children's ways of knowing and representing; rather, it is a clear call for creating opportunities for children to learn new, important mathematics in ways that make sense to them.

#6

Reader Reaction

We are interested in your reaction to the style and tone of Chapter 4. How might the chapter be more helpful?

What is your reaction to the section on "Mathematics Education Before Kindergarten?"

Are we clear enough about our expections for pre-K–2 children in the area of number? What additional guidance would be helpful?

Standard 1: Number and Operation

Mathematics instructional programs should foster the development of number and operation sense so that all students—

- **understand numbers, ways of representing numbers, relationships among numbers, and number systems;**
- **understand the meaning of operations and how they relate to each other;**
- **use computational tools and strategies fluently and estimate appropriately.**

Elaboration: Grades Pre-K–2

Number and operations are a major emphasis in grades pre-K–2 because understanding number is fundamental to all aspects of mathematics. Children come to school with rich and varied informal knowledge of number. A major goal is to build on this informal base toward more sophisticated understandings and skills. Children move from developing basic counting techniques to understanding number size and relationships, place value, and operations. They initially use their own representations of number and are introduced to more conventional notation. By the end of this grade band, children should be fluent (efficient and accurate) with the basic addition and subtraction facts and should be solving problems involving two- and three-digit numbers using strategies they have developed based on this knowledge.

As they solve new problems using a variety of approaches in school and play situations, children develop networks of mathematical connections. Increasing fluency with number computation supports children's data investigations, while concepts related to pattern support their skills in counting by twos, fives, and tens. Through experiences with shape, space, and number children begin to develop estimation skills related to quantity (e.g., how big is 100?) and size (e.g., how long is my foot?). Computational estimation strategies, including rounding, are addressed in later years.

Focus Areas for Grades Pre-K–2

- **Understand numbers, ways of representing numbers, relationships among numbers, and number systems**

  In grades pre-K–2, all students should—

  - count fluently with understanding and recognize "how many" in small sets of objects;
  - understand the cardinal and ordinal meaning of numbers in quantifying, measuring, and identifying the order of objects;
  - connect number words, the quantities they represent, numerals, and written words and represent numerical situations with each of these;
  - develop an understanding of the relative magnitude of numbers and make connections between the size of cardinal numbers and the counting sequence;
  - develop an understanding of the multiple relationships among whole numbers by comparing, ordering, estimating, composing, decomposing, and grouping number, including beginning understandings of place value;
  - understand and represent familiar fractions, such as 1/2 and 3/4.

- **Understand the meaning of operations and how they relate to each other**

  In grades pre-K–2, all students should—

  - understand different meanings of addition and subtraction of whole numbers and the relation between the two operations;
  - understand situations that lead to multiplication and division, such as equal groupings of objects and sharing equally;
  - develop understanding about the effects of the operations on whole numbers.

- **Use computational tools and strategies fluently and estimate appropriately**

In grades pre-K–2, all students should—

- develop and use strategies and algorithms to solve number problems;
- develop fluency with addition and subtraction facts by the end of second grade;
- compute using a variety of methods, including mental computation, paper and pencil, and calculators and choose an appropriate method for the situation;
- recognize whether numerical solutions are reasonable.

Discussion

• Number Concepts and Their Properties

Young children are motivated to count everything from treats to stairs they are climbing. From these beginnings, they develop increasingly powerful counting concepts and strategies. They learn to associate number words with collections of objects counted, establishing one-to-one correspondence by moving, touching, or pointing to objects as they say the number words. They learn that the last number named represents not only the last object, but also the total number of objects in the collection. They learn that counting objects in different order does not alter the result. Primary grade children benefit from many varied opportunities to develop, use, and practice counting as they quantify, measure, identify location, and solve problems.

Preschoolers should develop the fundamental idea that the next number in the counting sequence is one more than the number just named. They connect cardinality with ordinality (number sequence) by comparing the results of counting. For example, a child might say that one group of six items has "more" than a group of five items because "6 comes after 5." Kindergarten teachers should use every opportunity to develop number concepts by posing questions such as, "How many pencils are in the pencil can?" "Shall we count how many steps to the playground?" "Who is third in line?" and "How many of you want me to read this book about whales and how many of you want me to read this book about seals?"

In the early primary grades, students learn different counting strategies—such as skip counting by twos, fives, and tens or counting backwards—building a basis for subtraction and multiplication. Young children especially enjoy counting large numbers. Although counting is a major way to quantify, it is not the only one. Children also can look at a small group of objects (about six items or fewer) and recognize "how many." This is a skill called *subitizing*. As children develop this skill, they begin to use grouping strategies to count larger numbers. This ability to recognize "how many" supports the development of strategies for estimation as students visualize small groups within a larger group. For example, a second grader might report, "There are about five in each place. I circled groups of five in my head and then counted six groups. So, there are thirty in all!" This kind of thinking is part of number sense.

Number sense develops as children come to understand the size of numbers, develop multiple ways of thinking about and representing numbers, use numbers as referents and develop accurate perceptions about the effect of operations on numbers (Sowder 1992). They can estimate about how many cubes they can hold in their hand if they know a classmate can hold nine cubes. When the teacher asked the class if 19 + 18 would be more or less than 40, a student replied, "It's less than 40, because both numbers are less than 20 and 20 + 20 is 40."

In the early primary grades, children also develop the ability to mentally visualize numbers and/or to think about numbers without having a physical model. For example, a first grade teacher might show a child four blocks and a cloth hiding other blocks. Telling the child that there are seven objects in all, the teacher asks the child how many are hidden. Some children insist on raising the cloth before they will count. They need objects they can see to count. Others will be able to count only if they point at the cloth and form an image of how many motions it takes them to get to seven. Some may be able to see the four visible blocks and say, "Five, six, seven…so there are three hidden!"

Children gradually develop part-part-whole concepts. For example, in a situation where there are 4 red balls and 3 blue balls, three and four are parts and seven is the whole. Children can solve problems such as "27 + __ = 36" by starting at 27 and counting up to 36, keeping track of the 9 counts. Then, if they are asked to solve "36 – 9 = __," they may say immediately, "27." If asked how they know, they might respond, "Because we just did it!" The children understand that "27" and "9" are numbers in their own right, as well as two parts that make up the whole, "36." They also understand that subtraction is the inverse of addition (Steffe and Cobb 1988).

For some children who are still constructing meanings for individual numbers, there are many ideas to learn on the way to full understanding of place value. Children need to learn that tens are special "bridges" in counting ("…38, 39…40" or "...78, 79, 80"). Young children enjoy counting into the hundreds and come to appreciate important patterns in the numeration system through such experiences. Children also need to understand "10" as a special "unit of units," and recognize the word *ten* as a single entity (1 ten) and, at the same time, ten separate units (10 ones) (Cobb and Wheatley 1988). A hallmark of number sense is flexibility in thinking about numbers. This is critical as students develop understanding of the base-ten number system. Experiences modeling and naming numbers in different ways contribute to children's development of number sense, as illustrated in figure 4.1.

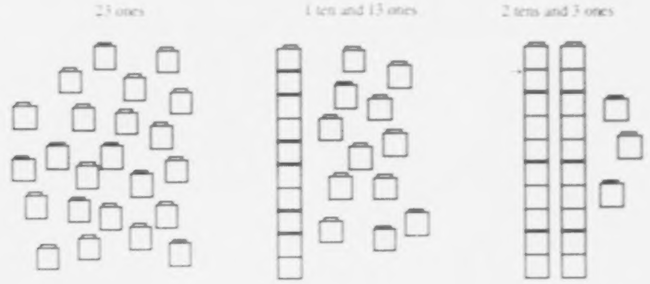


Figure 4.1. Counting models.

Concrete models, such as connecting cubes, sticks, or beans, can help students represent numbers in groups of ten and make the translation to the written symbols more meaningful. Gradually, children develop the flexibility to group and ungroup by tens. For example, they can express "23" as twenty-three ones (units), one ten plus thirteen, or two tens and three ones. Using materials in a rote manner does not support understanding. Some children, for example, might interpret the two tens and three ones in figure 4.1 as merely five blocks.

Technology can help children develop number sense through computer manipulatives. For example, students can break computer base-ten blocks into ones or glue ones together to form tens. The computer also links the blocks to symbols. For example, the number represented by the base-ten blocks is usually dynamically linked to the students' actions on the blocks. When the student changes the blocks arrangement, the number displayed is automatically changed as well. This visualization can help students make sense of their activity and the numbers.

Calculators also can be a useful tool in place value instruction, because seeing each value displayed on the calculator helps develop important insight into what digits are changing and when. For example, when students use the automatic constant to count by ones, they observe that the digit on the right changes every time they press the "+" key, but the digit in the tens place changes less frequently. Classroom conversations about such patterns focus children's attention on place value. Figure 4.2 shows a sample calculator activity.

Make a New Number

Start with 178. Do not press clear. Change to:	What keys did you press to reach the number?
278	
248	
208	
318	
419	

Figure 4.2. Calculator place value activity.

Young children in this grade span begin to develop concepts of fractions, starting with the common language they bring to the classroom, such as "one-half." Learning about fractions is especially effective when it is connected to everyday situations and meaningful problems. Although fractions are not a main emphasis in grades pre-K–2, informal, integrated experiences at this age will help develop a foundation for deeper learning in later grades. At this level, it is more important for students to recognize three parts of four equal parts or three-fourths of a folded paper that has been shaded, and to understand the language (*fourths means four equal parts of a whole*) than to memorize fraction notation.

- **Operations and Their Properties**

Many varied contexts are necessary for young children to develop understanding of the meanings of operations such as addition and subtraction. These contexts may arise through child-initiated activities, such as "How many books do we have if one child brings three books to the table and another child brings two books?" Other contexts include teacher-created stories. Children can solve simple joining and take-away problems by directly modeling the situation with objects or by using counting strategies, such as counting on. Children also benefit from opportunities to solve "missing addend" problems that arise from story or real situations.

Similarly, addition of equal groups is a meaning of multiplication that children also should explore through problem situations. They can investigate division, usually as sharing equally, with real objects and through story problems. Meanings are closely related to strategies used by children to solve problems. Most children naturally encounter and find ways to solve a wide variety of problems, regardless of whether they can use the symbols or language that older students associate with addition, subtraction, multiplication or division. They also begin to use properties such as commutativity and distributivity naturally. Use of these properties is a signal of young students' growing number sense.

In learning to solve problems, children act out situations, model problems with concrete materials, or represent them with drawings. Students at this level should learn to communicate their strategies for solving problems first orally and then in some kind of written form. Talking and writing about their thinking has two important benefits for children. It helps them to solidify their ideas, and it allows other children to hear different ways of understanding and solving problems. During the pre-K–2 years, children will learn to use conventional notation as well as their own recording systems.

- **Computation and Estimation**

Although young children often first compute by using objects, they gradually shift to doing many computations mentally or with the help of paper and pencil to record their thinking. As they explore numbers and operations on numbers, children develop strategies for figuring out and remembering basic arithmetic facts. These strategies, when used with the basic facts, give children ways to compute with larger numbers. For example, a child may compute 8 + 7 as 8 + 2 is 10 and 5 more is 15, based on his or her knowledge about 10. Then, to find 28 + 7, that child might apply the same basic strategy: 28 + 2 is 30 and 5 more is 35. As children work with larger numbers, their invented algorithms play an important role in linking between their informal, intuitive knowledge and more formal arithmetic. Research provides convincing evidence that children can and do invent their own computational strategies and that such inventions contribute to their mathematical development (Fennema et al. 1996; Fuson et al. 1997; Steffe 1994). In one longitudinal study, ninety percent of young children used invented strategies. Moreover, students who used invented strategies *before* they learned standard algorithms demonstrated better knowledge of base-ten concepts and could better extend their knowledge to new situations, such as making change for a $1.76 purchase from $3.00 (Carpenter et al. 1998).

The kinds of strategies that young students use are varied, and all students benefit from hearing about and reflecting on different approaches. For example, the following problem was posed to a group of second graders:

> We have 153 students at our school. There are 273 students at the school down the street. How many students are in both schools?

One student's work follows:

153+273
300 5 tens +7 tens
3+3=6
426 students

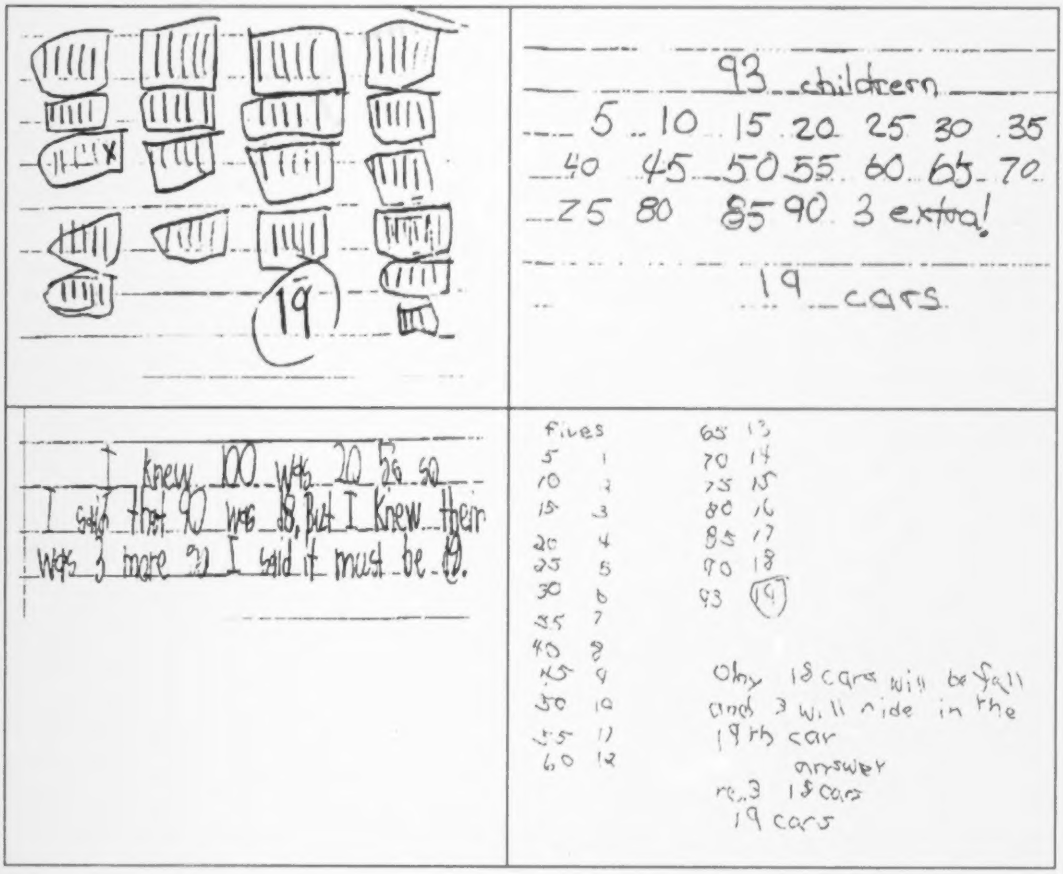
Another student modeled the problem with bean sticks that the class had made earlier in the year. The student placed hundred rafts, ten sticks, and loose beans on the table and then combined and traded to find the solution. A third student wrote nothing down and declared the answer based on mental computation. When asked to explain her answer, the third child said, "Well, 2 hundreds and 1 hundred are 3 hundreds, and 5 tens and 5 tens are 10 tens or another hundred, so that's 4 hundreds. There's still 2 tens left over, and 3 and 3 is 6, so it's 426." Another student used a conventional computation algorithm. While some students reached solutions more quickly than others, all students were able to develop and utilize a strategy to answer the question. In addition, the strategies that students used revealed to the teacher their level of fluency and level of thinking and provided an opportunity to gauge the need for additional attention and work.

A certain amount of practice is necessary to develop fluency with both basic fact recall and computation strategies for multi-digit numbers. Anderson, Reder, and Simon (1996) point out that practice is clearly essential for acquiring cognitive skills of almost any kind. Practice needs to be motivating and systematic if students are to develop fluency in computation, whether mentally, with manipulative materials, or with paper and pencil. Practice can be conducted in the context of other activities, including games that require computation as part of score-keeping, questions that emerge as children's literature is read and discussed, or focused activities that are part of another mathematical topic investigation (e.g., a problem to explore probability notions in which students roll two dice twenty times and keep a record of the sums of the two dice for each roll).

In grades pre-K–2, instructional emphasis should be on supporting students as they develop their own strategies for computation. In later grades, students will be introduced to more conventional computational algorithms. In the primary grades, if students can use their own methods efficiently and accurately, can explain why and how those methods work, and can extend their methods to new situations—such as different contexts or larger numbers—then the students have acceptable computational competence.

Since children within any class will use a variety of methods, the teacher's responsibility is to understand how children are thinking about various problems and encourage students to share how they think about numbers and how they solve problems. Teachers also must decide what tasks might challenge students and encourage them to construct more sophisticated strategies. Consider the opportunities for discussion offered by second graders' responses (shown in figure 4.3) to the following problem:

> There are 93 children going to the circus. In each car, 5 children can ride. How many cars will be needed?



I knew 100 was 20 5s so I said that 90 was 18, But I Knew their was 3 more so I said it must be 19.

Olny 18 cars will be fall and 3 will ride in the 19th car
answer
re.3 18 cars
19 cars

Figure 4.3. Children's computation strategies.

In the past, the main goal of arithmetic instruction was to develop skill in the efficient use of paper and pencil algorithms. Conventional computational algorithms are important in that they serve as succinct and efficient strategies. However, teachers need not introduce and develop them until later in the elementary grades. At the pre-K–2 level, children should develop strategies for computing that make sense to them and that are efficient and accurate. Often such strategies will be close to conventional algorithms. With opportunities to reflect on their own ways of thinking and the thinking of others, children of this age are building a foundation for mathematics that is meaningful. Students see that there are many ways to get

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an answer to a problem. Through class discussions, students can compare various strategies for ease of use, efficiency, and ease of explanation. As students encounter problem situations where computations become more cumbersome, they should be encouraged to use calculators to aid in problem solving.

The pre-K–2 years are significant for children in their development of number and operation sense. The movement from very simple demonstrations of number by the small child who holds up two fingers in response to the question, "How many is two?" to the second grader who solves sophisticated problems using multi-digit computation strategies represents dramatic growth in mathematical understandings. Number and operation concepts and skills are a primary focus of mathematics instruction during these years. When children have a broad sense of number, a strong beginning understanding of place value, and a working knowledge of the effects of operations on numbers, they are prepared for new challenges in grades 3-5.

Standard 2: Patterns, Functions, and Algebra

Mathematics instructional programs should include attention to patterns, functions, symbols, and models so that all students—

- **understand various types of patterns and functional relationships;**
- **use symbolic forms to represent and analyze mathematical situations and structures;**
- **use mathematical models and analyze change in both real and abstract contexts.**

Elaboration: Grades Pre-K–2

Children develop beginning concepts related to patterns, functions, and algebra at an early age. Even before formal schooling, recognition of patterns and comparisons are important components of a child's intellectual development. Children's observations and discussions of how things change will eventually lead to the notion of function. Children's representations of mathematical situations using concrete objects and pictures and their growing familiarity with symbols are the roots of symbolic representations that are essential in later grades.

Such concepts evolve and mature over the years. In the primary grades, they are manifested through work with classification, pattern and number, and using language and notation that is appropriate for young children. Many of the learning goals in the Patterns, Functions, and Algebra content area are achieved while children are working with number and operation. Other goals develop as children increasingly use mathematics to organize the world around them.

Major learning goals in the primary grades in the Patterns, Functions, and Algebra content area include the ability to sort and classify, analyze simple patterns and make predictions about them, describe change, represent situations mathematically, and develop an intuitive understanding and use of properties of numbers and operations.

Standard 2 Focus Areas for Grades Pre-K–2

- **Understand various types of patterns and functional relationships**

  In grades pre-K–2, all students should—

  - sort and classify objects by different properties;
  - order objects by size or other numerical property (seriation);
  - identify, analyze, and extend patterns and recognize the same pattern in different manifestations;
  - describe how both repeating and growing patterns are generated.

- **Use symbolic forms to represent and analyze mathematical situations and structures**

  In grades pre-K–2, all students should—

  - illustrate general principles (e.g., commutativity) using specific numbers;
  - understand reversal of operation;
  - use concrete, pictorial, and verbal representations of numerical situations, including invented notation;
  - use appropriate symbolic representation of mathematical situations.

- **Use mathematical models and analyze change in both real and abstract contexts**

  In grades pre-K–2, all students should—

  - make comparisons and describe change qualitatively (e.g., taller than);
  - make comparisons and describe change quantitatively (e.g., 3 inches taller);
  - model concrete situations using addition and subtraction of whole numbers.

Discussion

- **Patterns, Functions, and Relations**

When young children are given a bag of assorted stickers, they quickly notice many differences among the items in the set. They can sort the stickers into groups having similar traits like such as color, size or design. These experiences develop the foundation for many ideas of mathematics. Sorting and classifying are important aspects of young children's mathematical development so they are discussed not only in this section, but also in the section on Data Analysis, Statistics, and Probability and in the section on Reasoning and Proof.

Young children notice regular arrangements of objects, numbers, and shapes and can extend these patterns, using them to make predictions. Patterns can be found outside of mathematics lessons. Preschoolers recognize patterns in their environment on fences, walkways, and on buildings. They use these patterns to predict what would come next when the sequence is extended. Patterns help children develop the ability to form generalizations. For example, children can recognize that the color pattern "blue, blue, red, blue, blue, red..." is the same inform as "clap, clap, stomp, clap, clap, stomp...." This recognition lays the foundation for the notion that two things of very different nature can be the same in some important ways.

Patterns are a way for children to organize their world. They know that, for example, "first comes breakfast, then school..." and "Monday we go to art, Tuesday we go to gym...." Patterns allow children to recognize order and to predict. Children who see the digits, "1, 2, 3, 4, 5, 6, 7, 8, 9" repeated over and over will see a pattern that helps them with the task of learning to count to 100—a formidable task for students who do not recognize the pattern.

Patterns are also a precursor to function, and it is important for children to learn to record numerical relations in tables. For instance, if one fruit roll-up costs 10¢, and 2 cost 20¢, then how much would 3 fruit roll-ups cost? How much would 7 cost? Within the pattern there is the seed of the general function rule, "the cost of *n* is 10*n*," as well as the notion that every input can be associated with an output.

Attribute blocks and computer environments enable children to create more complex patterns using more than one attribute. Materials such as a Hundred Chart can provide a visual way of highlighting various patterns in numbers (see figure 4.4). A group of first grade students were skip counting by twos and coloring the corresponding numbers in a hundred chart. "I know how this goes," said Dennis as he proceeded to fill in the pattern without counting. "Color one, skip one, color one, skip one." "Wait," said Angie, "we could just color this whole stripe down."

1	**2**	3	**4**	5	**6**	7	**8**	9	**10**
11	**12**	13	**14**	15	**16**	17	**18**	19	**20**
21	**22**	23	**24**	25	26	27	28	29	30
31	**32**	33	**34**	35	36	37	38	39	40
41	**42**	43	**44**	45	46	47	48	49	50
51	**52**	53	**54**	55	56	57	58	59	60
61	**62**	63	**64**	65	66	67	68	69	70
71	**72**	73	**74**	75	76	77	78	79	80
81	**82**	83	**84**	85	86	87	88	89	90
91	**92**	93	**94**	95	96	97	98	99	100

Figure 4.4. Patterns on a hundred chart.

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- **Structure and Symbolic Notation**

As children have opportunities to generalize from observations about number and operations, they are forming the basis of algebraic thinking. For example, adding pairs of numbers can lead children to infer that when two numbers are added, the order does not matter. Their method of verification might be to count all the objects in each group, rearrange them, and count again. For example, 3 cubes plus 5 cubes gives the same answer as 5 cubes plus 3 cubes. This method could be applied with any numbers, so that children do not have to model concretely for every pair. Similarly, when children break apart numbers and rearrange them (such as adding 8 + 5 and thinking "8 + 2 is 10 and 3 more is 13"), they are applying the associative property of addition. While it is not essential to introduce vocabulary such as "commutativity" or "associativity" at this age, it is important for the teacher to be aware of the algebraic properties used by children and to help students develop an awareness of what they are doing.

The notion of inverse and the ability to perform multi-step processes also are important in later mathematical learning. Thus, young students should work with tools and situations where processes can be reversed and where operations or other actions are done sequentially. For example, a primary grade teacher showed her students a card numbered "11" and said "I've written something on the back." She turned the card over to show the number "12." Then she showed the students a second card with "12" on the front and "13" on the back. As she continued showing students the cards, each time she asked the students, "What do you think will be on the back?" After the students figured out that she was adding one, the teacher brought out a different set of cards. These were also numbered front and back, but with "31" and "33," "32" and "34," "33" and "35," and so on. Soon the children figured out that the numbers on the backs of the cards were two more than the numbers on the fronts. They wanted to play again. "But," said the teacher, "we can't do that until I make another set of cards." One child spoke up and said, "You don't have to do that, we can just flip the cards over. The cards will all be minus 2."

Children's spontaneous efforts to describe mathematical situations can be concrete, pictorial, or verbal and can involve their own invented notation and symbolism. Consider the example from a kindergarten class shown in figure 4.5. This was the child's invented notation for the number ten and a half.

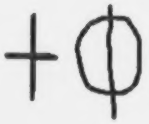

Figure 4.5. A child's notation of 10 1/2

Children's ability to use symbolic notation should be encouraged by activities involving manipulation of objects and quantities that symbols represent. Through classroom discussions of different representations, children's use of symbols becomes more detailed and mathematically richer. Teachers may scaffold children's explanations by scribing until children are proficient enough with writing to record their ideas. Original representations also will remain important throughout the child's mathematical study, and they should be encouraged. Conventional symbolic representation and manipulation begins to be an important part of the child's mathematical activity toward the end of this grade band.

- **Change and Modeling**

With teachers' guidance, children learn to make mathematical models to represent and solve problems. For example, a teacher may pose the problem: "There are six children. How many eyes are there?" One child will draw six faces and count the eyes one at a time. Another will draw six circles, put two cubes on each circle, and then count. A child might represent the problem as "2 + 2 + 2 + 2 + 2 + 2 = 12."

When children measure something over time, they can describe change both qualitatively (e.g., "Today is colder than yesterday") and quantitatively (e.g., "I am 2 inches taller than I was a year ago"). Some changes are predictable. For instance, children grow taller, not shorter, as they get older. The notions that things can change over time, that many such changes can be described mathematically, and that many of changes are predictable are basic understandings that will later support student's work in this area.

Recognizing and using patterns and learning to represent situations mathematically are important aspects of pre-K–2 mathematics. Children use these understandings, which are an important aspect of the development of number sense, to explore number, space, measurement and data.

Standard 3: Geometry and Spatial Sense

Mathematics instructional programs should include attention to geometry and spatial sense so that all students—

- **analyze characteristics and properties of two- and three-dimensional geometric objects;**
- **select and use different representational systems, including coordinate geometry and graph theory;**
- **recognize the usefulness of transformations and symmetry in analyzing mathematical situations;**
- **use visualization and spatial reasoning to solve problems both within and outside of mathematics.**

Elaboration: Grades Pre-K–2

The development of geometric and spatial knowledge begins before schooling and is stimulated by exploration of shapes and structures in the child's environment. Geometric ideas help children systematically represent and describe their world. They learn to represent two- and three-dimensional shapes through drawings, block constructions, dramatizations, and verbal language. Children use knowledge about direction and position early in their spoken language and locate objects by giving and following directions using this language. For example, young children delight in traveling over, under, around, and through objects. Their geometric and spatial capabilities frequently exceed their numerical skills, so building on these strengths fosters enthusiasm for mathematics while providing a context to develop number concepts.

Focus Areas for Grades Pre-K–2

- **Analyze characteristics and properties of two- and three-dimensional geometric objects**

  In grades pre-K–2, all students should—

  - recognize, name, build, draw, describe, compare, and sort two- and three-dimensional shapes;
  - recognize and locate geometric shapes and structures in the world;
  - describe attributes and parts of two- and three-dimensional shapes;
  - investigate and predict the results of putting together and taking apart shapes;
  - recognize congruent and similar shapes;
  - relate geometric ideas to number and measurement ideas.

- **Select and use different representational systems, including coordinate geometry and graph theory**

  In grades pre-K–2, all students should—

  - describe, name, interpret, and apply ideas of relative position in space;
  - describe, name, interpret, and apply ideas of direction and distance in navigating space;
  - find and name locations with simple relations (e.g., near to) and coordinate systems (maps).

- **Recognize the usefulness of transformations and symmetry in analyzing mathematical situations**

  In grades pre-K–2, all students should—

  - recognize and apply slides, flips, turns; predict the effects of transformations on shapes;
  - recognize and create reflectional and rotational symmetry of two- and three-dimensional objects.

- **Use visualization and spatial reasoning to solve problems both within and outside of mathematics**

In grades pre-K–2, all students should—

- create mental images of geometric shapes (spatial memory and spatial visualization);
- determine and represent objects from different perspectives and points of view;
- recognize and describe spatial relationships.

Discussion

- **Shape and Structure**

Children begin forming concepts of shape in the years before they begin formal schooling. Some of these ideas stabilize as early as age six. For example, children of this age can easily recognize circles and squares. Thus, the early primary grades are an ideal time to build on this knowledge and help children to refine and extend their understanding.

Children first learn to recognize shapes as wholes. They may believe that a given figure is a rectangle because "it looks like a door." They represent shapes in drawings and block constructions and through dramatizations using their own language. They search for examples three-dimensional shapes such as spheres, cylinders, and "boxes," and two-dimensional shapes such as circles, rectangles, and triangles. Children enjoy combining, folding, and creating new shapes or copying shapes on geoboards or geodot paper. Children sort building blocks as they put them away on shelves. They can challenge themselves by feeling a geometric object inside a bag and identifying it or matching it with a corresponding set of objects.

An important part of pre-K–2 geometry strand is the naming and describing of shapes. Young children should begin by using their own vocabulary as they need it and gradually should incorporate conventional terminology as the teacher uses such terms. Children recognize and characterize shapes by their attributes or properties. For instance, a student might think of a square as a figure that has four sides of equal length and four right angles. Children should eventually understand that words like "straight" and "similar" have specific, precise mathematical definitions, but terminology should not be the focus of the pre-K–2 geometry program. It is more important that these early experiences with geometry lay a solid foundation for more formal work in of geometry in later grades.

Well-chosen materials and appropriately structured environments encourage children to explore attributes and properties of shapes. For example, young children can experiment with a set of pattern blocks. They can compare the different shaped blocks in the set by identifying their similarities and differences. They can compare the size of the blocks (e.g., length of sides or size of angles) by placing one shape on top of another. Children also can investigate congruence by combining blocks to form new shapes that match other blocks. They might sort shapes by their attributes (all those with a square corner here) or play "guess my shape" from attribute clues. Pattern blocks and models of geometric solids provide a rich variety of opportunities for classification.

Commonly available materials can provide students the concrete representations they may need to understand and learn about attributes of two-dimensional shapes. Toothpicks and marshmallows can be used to build polygons (toothpicks for the sides and marshmallows for the corners). Construction paper squares can be folded in ways that demonstrate equality of sides or angles or to find lines of symmetry .

Children should explore geometric properties by combining familiar figures to form new shapes. Similarly, they should cut such shapes apart to form other shapes (see figure 4.6). Computer "shape" environments can provide particularly rich experiences in putting together or taking apart (composing and decomposing) shapes, because shapes can be positioned apart or together easily.

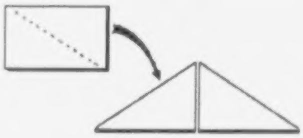

Figure 4.6. Breaking apart and putting together shapes.

Attribute and tangram puzzles engage children in interesting geometric problem-solving experiences. Such activities build ideas in patterning, functions, and pre-algebra. They also develop awareness and intuition about the *structures* of shapes and space. Children learn that certain shapes tile the plane, or "fit together with no holes."

Critical links between the worlds of geometry and number are forged as children count the sides and corners of shapes, describe shapes seen in patterns ("3 triangles, 4 squares, 3 triangles, 4 squares..."), and measure the sides of shapes ("It's a square mat—the sides are all the same length!"). Modeling various arrangements of numbers with geometric shapes allows children to visualize numbers geometrically. For example, children can make different arrangements of "6." One child may choose 3 red circles and 3 yellow squares or, perhaps, 1 red circle and 5 yellow squares. When children use colored connecting cubes to model numbers, they are developing the concept of a number line.

Children's conversations and pictures are the primary sources for examining children's developing geometric knowledge. When students draw and write about their experiences, they provide pictorial representations and recorded reflections about what they understand. The teacher can use this information for assessment or to guide instructional planning.

- **Coordinate Geometry and Other Geometric Models**

Early childhood teachers extend children's knowledge of relative position in space through many conversations during the school day. For example, when children act out the story of *The Three Billy Goats Gruff*, they illustrate "over" and "under," "large" and "small," and "near" and "far." Through such experiences, children learn much about location, space, and shape. In addition, four types of mathematical questions regarding navigation and maps help children develop a variety of spatial understandings. These are: Direction—which way?, distance—how far?, location—where?, and representation—what objects? To answer these questions, students need to develop a variety of skills.

Navigation activities emphasize direction and distance. After young children master environmental *directions* and relative positions such as "above," "over," and "behind," they can develop navigation ideas such as "left," "right," and "front," and global directions such as "north," "south," "east," and "west." These ideas, along with the concepts of *distance* and *measurement*, might be developed as children build and read maps of their own environments. For example, children might plan to build a three-dimensional model of their classroom and discuss which solids (such as building blocks) might be used to represent various objects like a desk or a file cabinet. Later, children might mark paths, as from a table to the wastebasket, with masking tape, thus emphasizing the path's continuity. With the teacher, children could draw a map of the room that includes the path. Some teachers have engaged children in making maps by gluing photographs or pictures from magazines to a large sheet of paper.

In a similar exercise, children might "walk out" shapes. After they walk around a rectangular shaped rug, for example, the children could command the computer "turtle" to "walk" a rectangle on the computer screen. Some children might measure the rug with footprints and create a computer-generated rectangle with the same dimensions—an early example of scaling and similarity. Some computer programs allow children to navigate through mazes or maps. Using these programs, children can learn orientation, direction, perspective, and measurement concepts. The computer experiences help them abstract, generalize, and symbolize their experiences of navigating.

• Transformations and Symmetry

Young children move shapes intuitively, as when they solve a tangram puzzle. They turn tangram pieces, flip them over, and experiment with new arrangements. These actions are explorations with transformations, and they are an important part of spatial learning. Young children can learn substantial mathematics through guided explorations. For example, working with structured geometric materials such as pattern blocks, children create designs with linear and rotational symmetry. A teacher may challenge students to recognize, describe, and informally prove the symmetric characteristics of their designs.

At this age, children naturally use their own physical experiences and the geometric patterns they create while playing with shapes to learn about symmetries and transformations such as slides, turns, and flips. Folding paper cutouts or using mirrors to investigate lines and planes of symmetry are other ways for children to observe figures in a variety of positions, become aware of their important properties, and compare and contrast them.

Children investigate the effects of various transformations on shapes. They learn that slides, flips, and turns change a shape's position or orientation, but not its size or shape. As children fill puzzles with geometric shapes, they learn to change the orientation of shapes without changing their size. By creating similar shapes on the geoboard children can demonstrate change in size without change in shape. figure 4.7 shows an exercise in which children hold a long loop of yarn so that each hand serves as a vertex. In this arrangement, children can explore the effect of changing the size of an angle, or increasing the number of sides while the perimeter is unchanged. With technology, children can think about the motions they are using. For example, children using shapes programs often have to consciously choose a motion. This helps them abstract ideas about sliding, flipping, and turning. A variety of experiences with movement strengthens children's knowledge of shape and structure, as well as their spatial skills.

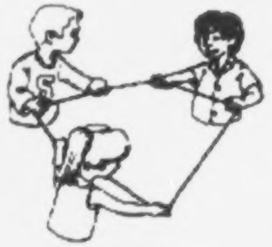

Figure 4.7. Polygons with a string.

- **Visualization and Spatial Reasoning**

Spatial visualization includes building and manipulating mental representations of shapes, relationships, and transformations. Work with concrete shapes, such as tangrams, attribute and pattern blocks, and other shape sets, provides a valuable foundation for spatial sense. To further develop students' abilities, teachers might ask them to see in their "minds' eye" the shapes that would result from flipping a certain shape or from cutting a square from corner to corner. Thus, many of the shape and transformation activities will build spatial reasoning if children are asked to imagine and predict.

Spatial memory is employed, for example, when children recall the configuration of the dots on the dominoes and know at once number without counting the dots. Classroom activities can be designed specifically to enhance visualization. When a number of objects (such as scissors, a pen, a leaf, a paper clip, and a block) are placed on the overhead, the teacher can show the objects briefly. Children are then asked to name the objects they glimpsed. Or, the teacher can have students close their eyes, take one object away, and ask the children what was removed. In another "quick image" activity (Yackel and Wheatley 1990), children briefly see the simple configuration shown in figure 4.8 on an overhead projector. Then, they try to reproduce it. The configuration is shown again for a couple of seconds, and children are encouraged to modify their drawings. The process may be repeated several times so that children have opportunities to evaluate and self-correct their work. During a discussion, teachers should ask children how they approached the task. Different explanations include: "three triangles," "a sailboat sinking," "a square with two lines through it," "a 'y' in a box," and "a sandwich that has been cut into three pieces."

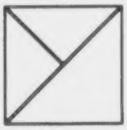

Figure 4.8. Quick image activity.

Navigation activities also develop spatial reasoning. Children might be asked, for example, to visualize and describe the path they just walked from the library. Or, as children build a model of the classroom with building blocks, they can be asked to view it from different *perspectives*. They might identify block structures from various viewpoints, matching views of the same structure that are portrayed from different perspectives, or try to find the viewpoint from which a photograph was taken.

In summary, geometry activities provide children a perspective on mathematical thinking different from, but connected to, the world of numbers. As children develop facility in shape, structure, location, transformations, and spatial reasoning, they are laying the foundation for understanding not only their spatial world, but for understanding other topics in mathematics and in subjects such as art, science, and social studies.

Standard 4: Measurement

Mathematics instructional programs should include attention to measurement so that all students—

- **understand attributes, units, and systems of measurement;**
- **apply a variety of techniques, tools, and formulas for determining measurements.**

Elaboration: Grades Pre-K–2

Measurement is one of the most widely used applications of mathematics. The teaching of measurement should build on children's intuitive spatial understanding and help children establish connections between spatial understanding and number. Children need to understand the attribute to be measured as well as what it means to measure. But, before they are capable of such understanding, young children must first experience many different activities that involve informal measurement, visual comparison, and manipulation of concrete materials. These experiences should focus on comparing objects directly, covering them with various units, and counting the units. A firm foundation in measurement concepts will eventually enable children to use measurement systems, tools, and techniques.

Focus Areas for Grades Pre-K–2

- **Understand attributes, units, and systems of measurement**

  In grades pre-K–2, all students should—
  - recognize the attributes of length, capacity, weight, area, and time;
  - compare and order objects qualitatively by these attributes;
  - make and use measurements in natural situations;
  - develop referents for estimation;
  - develop a sense of the unit (e.g., length, area) through estimation.

- **Apply a variety of techniques, tools, and formulas for determining measurements**

  In grades pre-K–2, all students should—
  - use tools such as rulers to measure;
  - measure with same size unit (nonstandard and standard);
  - use repetition of units (iteration) to measure length and area.

Discussion

- **Understand Attributes, Units, and Systems of Measure**

Children begin to develop an understanding of attributes through experiences in which they make decisions about the sizes of objects by looking, touching, or comparing objects directly. Young children should have a variety of experiences comparing the size of objects in various dimensions. For example, they can be asked to find all the objects in the room that are as long as their foot. Such comparisons arise naturally. When they do, teachers might join the debate about, say, whose block building is biggest by asking, "Do you mean the tallest (gesturing) like Marcia's building? Or do you mean the biggest covers more of the rug, as Lynn's building does?" Such dialogues provide an opportunity for students to build the vocabulary associated with measurement, including "tallest," "largest," and "same as."

Comparing and ordering objects provides students the skills to see relationships. Estimation activities should be integrated throughout measurement. For example, students should be able to find an object "about as tall" as they are. And, as they begin to use standard units, students should be able to locate an object of a given measure, such as finding something that is "about five centimeters long." The purpose of estimation activities is to help children understand the attributes and the process of measuring as well to promote an awareness of the sizes of units. It also supports the development of number concepts and skills. Everyday situations are important to consider. A teacher might ask, "What time do you get up? About how long does it take to get to school?" Because exact measurements are not always needed to answer such questions, children should realize that it is often appropriate to report a measurement as an estimation: "about three hours" or "between eight and nine centimeters." For example, an activity might involve measuring the length of a table using connecting cubes. Children might make a "train" of cubes as long as the worktable and break the train into lengths of ten to check their measurements. There will be partial sets of ten. With this information, children will discuss ways of efficiently "counting" 4 tens and 3 more units. This experience connects place value and measurement.

The process of measuring is identical for any attribute: choose a unit, compare that unit to the object, and report the number of units. The number of units can be determined by counting or by using a measurement tool. Children can tile a space and count the number of tiles required to find the area. For linear measurement, students can record their height by using a meter stick. Although most measurement tasks for early primary children will use nonstandard units, it is appropriate for those children to experiment with standard measures and to use many of them by the end of second grade.

Pre-K–2 students have opportunities to investigate area as they make designs with pattern block shapes. They frequently create designs that are made up of smaller, but related shapes. They might, for example, determine how many small triangles make up the large rectangle. Observant teachers use these situations to establish and extend notions of area.

At a young age, children develop beginning notions of capacity and volume. For example, they might pour sand or water from one container to another or count while filling a box with cubes. They intuitively know to compare the capacity of the various containers by counting the number of scoops required to fill each. They also experiment with filling smaller containers with the contents of larger ones. Teachers continue to guide children's experiences with capacity by the resources they make available, the number of opportunities they provide for their use and by engaging students in a discussion about what they are doing, why they are doing it, and the results of their actions. This type of discourse also builds students' conceptual and procedural knowledge of measurement.

It also is important for primary children to have experiences with weighing objects. Using balances allows children to develop an understanding of comparative weights. They can predict that two cubes will weigh the same as twenty links and test their conjecture. They can measure equal amounts of clay for an art project. Or, they can compare the weight of different size cookies. Activities such as these reinforce the attribute of weight and notions of equality that are so important for other strands of mathematics and other content areas.

The use of balances and scales permits students to assign numerical values to the weight of objects and to use standard measures in meaningful ways. In such environments, young children may use units of varying size, define and create their own units, maintain or predict unit size, and create length rather than end point representations through either iterative or numeric distance commands.

Children also need to explore concepts of time and develop ways to measure it. Students should recognize and develop a sense of the intervals of time between events and activities, such as the time between recess and the dismissal of school, between supper and bedtime, or between fall and winter. Opportunities arise throughout the school day for teachers to focus on time as a unit of measure through short conversations with their students. A teacher might say, for example, "It's one o'clock, time for gym!" "Look at the clock. It is just like the picture on our schedule" "Let's read for one more minute—how long is that?" and "Children, can you clean up before the three-minute sand timer runs out?" The emphasis at the pre-K–2 level should be on developing concepts of time more than the mechanics of using an analog clock. However, by the end of the grade band, children should be able to tell time using different timepieces.

- **Apply a Variety of Techniques and Tools**

If children's initial measurement explorations use a variety of units, nonstandard as well as standard, they will develop understanding about the nature of units and the need for standard units. For example, when students measure their height using a length of string and a meter stick and then compare uniformity of results, they build an awareness of the value of a standard unit. The same is true when different children tile an area using dominoes and unit tiles. In counting the number of objects used, the children recognize the discrepancies between measurements. Such experiences can lead to a discussion and application of standard tools. Worthwhile experiences include actually measuring with different materials, from paper clips and toothpicks to rulers.

Teachers cannot assume that children understand measurement fully, even when they can complete standard textbook exercises or worksheets involving reading a pictured ruler aligned with a pictured object. Children need hands-on experiences to develop both concepts and skills of measurement. These develop together as children place manipulative units without leaving spaces between them or compare the result of measuring with different nonstandard units. Similarly, with rulers, children learn concepts and procedures, including accurate alignment (ignoring the leading edge at the beginning of many rulers), starting at zero, and focusing on the lengths of the units rather than only the numbers on the ruler. The question, "What are you counting?" can not be overemphasized.

Because children are operating on quantities such as lengths—comparing them, putting them together, and taking them apart—measurement activities are also strong contributors to children's development of spatial sense. Also, combining or comparing measures using addition or subtraction provides another model for these arithmetic processes and contributes to the development of number and operation sense.

In summary, pre-K–2 students should learn the underlying concepts and process skills of measurement through actual experiences. As measurement is introduced, it should be integrated throughout mathematics and other curriculum areas. Because measurement bridges two main areas of mathematics, geometry and number, measurement activities can simultaneously teach important everyday skills and strengthen other important topics in mathematics. Not only will measurement experiences enhance other topics, they also will give children opportunities to develop and retain measurement concepts and skills that will be formalized and expanded in later years.

Standard 5: Data Analysis, Statistics, and Probability

Mathematics instructional programs should include attention to data analysis, statistics, and probability so that all students—

- **pose questions and collect, organize, and represent data to answer those questions;**
- **interpret data using methods of exploratory data analysis;**
- **develop and evaluate inferences, predictions, and arguments that are based on data;**
- **understand and apply basic notions of chance and probability.**

Elaboration: Grades Pre-K–2

Early experiences with data help children develop the concepts that underlie data analysis and statistics. Informal activities of comparing, sorting, and counting are the beginnings of mathematical understandings of data and data analysis. The types of activities needed and appropriate for preschoolers vary greatly from those for second graders; however, in all primary grades, students can pose questions to investigate, organize the responses, and create representations of their data. Through their data investigations, children develop concepts for informed decision making.

Inferring behavior about a population from a sample is taken up in grades 3-5. However, at the upper end of this grade band it is appropriate for children to discuss when conclusions about one population might be expected to be similar to conclusions about another population. Considerations like these are the precursors to understanding the notion of inferences from samples.

Notions about probability at this level are informal and focus on judgements that children make because of their experiences. The experiences that underlie experimental probability, such as tossing number cubes, occur at this grade band, but the primary purpose for these activities is focused on other strands, such as number.

Focus Areas for Grades Pre-K–2

- **Pose questions and collect, organize, and represent data to answer those questions**

  In grades pre-K–2, all students should—

  - gather data about themselves and their surroundings to answer questions that involve multiple responses;
  - sort and classify objects and organize data according to attributes of the objects;
  - represent data to convey results at a glance using concrete objects, pictures, and numbers.

- **Interpret data using methods of exploratory data analysis**

In grades pre-K–2, all students should—

- describe parts of the data and the data as a whole;
- identify parts of the data with special characteristics, for example the category with the most frequent response.

- **Develop and evaluate inferences, predictions, and arguments that are based on data**

- **Understand and apply basic notions of chance and probability**

In grades pre-K–2, all students should—

- understand notions such as certain, impossible, more likely, less likely.

Discussion

- **Collect, Organize, and Represent Data**

The primary purpose of collecting data is to answer questions when the answers are not immediately obvious. The natural inclination of children to ask questions must be nurtured. At the same time, students need to develop ways to gather information to answer their questions so that they learn when and how to make decisions based on data. As children enter school and their interests move from their immediate surroundings to broader environments, they begin to learn how to keep track of multiple responses to their questions and those posed by others. Children also begin to learn how to refine their questions to get the information they need.

Organizing data into categories begins long before formal schooling through informal sorting experiences, often under the direction of adults. For example, a parent might encourage their child to help put away groceries and provide direction such as, "These groceries go in the refrigerator; those go on the shelf." These informal experiences help develop the understanding of "things that go together." By talking about what goes with what, children build important vocabulary needed to describe attributes, classify accordingly and develop informal notions of grouping according to criteria. Such essential early learning experiences and language development together serve to focus children's attention on attributes of objects. By the end of the primary years, children are expected to identify multiple attributes.

Organizing and representing data can take place as data are gathered or afterward. If children are asked to put a counter into a cup to indicate whether they vote for story A or story B, the responses are organized as the data are gathered. In contrast, children surveying their classmates to determine the number of pockets on their clothing may gather the data by listing names and noting the number of pockets beside each name. Counting how many children have how many pockets (organizing the data) and coming up with a way to display this information are additional steps that follow data collection.

To address a particular question such as, "What snacks should be available in the school vending machine?" a variety of real objects such as packages of raisins, peanuts, or granola bars can be collected and used to poll students. At other times, pictures of objects, counters, namecards, or tallies can be used to indicate preferences. Children can learn how to gather and organize data as prompted by their localized interests and setting. Gradually, they come to understand that how they gather and organize the data depends upon the questions they are trying to answer.

Refining questions, considering possibilities, deciding how to collect information, and organizing data are not automatic; they are skills acquired through experience. In one class, for example, students became interested in the question of whether more families had cars with two doors or four doors. As they planned, the students had to decide if trucks should be included. What about vans with three doors or station wagons with five doors? After the class had settled on common categories, different groups of students kept track of the data in different ways. One group put cubes in different cups that represented the different categories. Another group recorded the data using tallies. A third group made a list without attempting to organize them and agree on the results of their data collection.

Children should create displays to communicate what they have found out and their representations should be valued because they reflect the child's understandings. As students' questions become more sophisticated and their samples larger, their use of traditional representations will increase. By the end of primary grades, children should organize and display their data through both graphical displays and numerical summaries. They should be able to use counts, tallies, tables, bar graphs, and line plots. Titles and labels should clearly identify what the data represent.

Scales used in displays is an important concept. For example, the tallest column in a display may not represent the most items if the pictures or symbols vary in size (see figure 4.9). In this case, the teacher has an opportunity to guide discussion about revising the graph on the left to be like the one on the right, so it conveys information accurately at a glance.

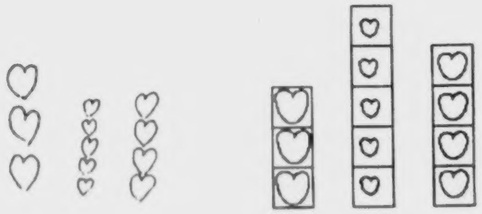

Figure 4.9. Displaying data.

• Analyze and Interpret Data

When data are displayed in an organized manner, children can describe the results of their investigations and answer the questions they have posed. Children gradually become better at forming questions which will get the information they want (i.e., most of their families' cars have four doors, or the least popular flavor of ice cream is strawberry). Older children can compare parts of the data and make statements about the data as a whole, such as "The same number of children have dogs as have cats" and "Most children have some kind of pet." By the end of this grade band, children begin to recognize appropriate conclusions about their data and to question inappropriate statements. For example, two students interested in how many of their classmates watched a particular television show surveyed only their friends. When they reported their results to the class, one student complained, "You didn't ask me and I watched it!" Another said, "Wait a minute, you didn't ask me and I *didn't* watch it. I bet most kids didn't watch it." In situations such as this, discussion among students is more helpful then adult directives, which may assume understandings that the children do not yet have.

The notion of a sample is a difficult one for young children. Most of their data gathering is for full populations, such as their own class. With guidance, students can begin to recognize when conclusions about one population do not necessarily apply to another. In one first grade class, for example, a teacher read *Whistle for Willie*. Afterwards, a discussion about whistling took place, followed by a survey. The students discovered that nineteen children in the class could not whistle and that eight could. When the teacher asked the class to title the chart they had created, the students agreed that an appropriate title would be, "Most children cannot whistle." The teacher then asked, "What do you think would happen if we asked the fourth graders?" The students repeated the survey and discovered that almost every fourth grader could whistle. One student said, "We have to change our title and say most first graders in our class cannot whistle."

- **Inference and Prediction**

Inference and prediction are more advanced aspects of this standard. Since inferences that go beyond the defined population at hand are based on probalistic thinking, the development of these concepts requires extensive work with sampling that comes in the next grade band. Informal discussions about whether or not children in the class next door would reach a similar conclusion are appropriate beginnings for making inferences.

- **Probability**

Answering questions about the likelihood of events, such as more likely or less likely, begins at this level. Young children enjoy thinking about impossible events, often encountering them in the books they are learning to read. Questions about more and less likely events should come from the students' experiences, and the answers will generally depend on the community and where it is located. The question, "Is it likely to snow tomorrow?" has different answers in Toronto and in San Diego.

The notion of probabilities as numbers is generally not appropriate at this age, since it depends on the idea of ratio. Interpreting experimental probabilities meaningfully is difficult for the same reason. Teachers can address the beginnings of experimental probability through informal activities that reinforce other content strands. For example, students repeatedly tossing two number cubes and adding the results of each toss may begin to keep track of the results. They realize that a sum of one is impossible, that a sum of two or twelve is rare, and that the sums six, seven, and eight are fairly common. Through discussion, they may realize that this has something to do with the number of ways you can get a particular sum from two number cubes, but the exact calculation of the probabilities will occur in later grades.

Standard 6: Problem Solving

Mathematics instructional programs should focus on solving problems as part of understanding mathematics so that all students—

- **build new mathematical knowledge through their work with problems;**
- **develop a disposition to formulate, represent, abstract, and generalize in situations within and outside mathematics;**
- **apply a wide variety of strategies to solve problems and adapt the strategies to new situations;**
- **monitor and reflect on their mathematical thinking in solving problems.**

Elaboration: Grades Pre-K–2

Problem solving is natural to small children, for whom much of the world is new and whose responses are often unconstrained by previous habit. The challenge in grades pre-K–2 is to help children's innate problem-solving inclinations develop so that they have a positive problem-solving disposition, develop and use a variety of problem-solving strategies, and incorporate habits of monitoring and adjusting. In both planned activities and spontaneous situations, problem solving should be firmly embedded in the mathematics curriculum, thus introducing, extending, and reinforcing children's mathematical knowledge.

Because young children in the same classroom may have very different understandings and skills, a problem situation for one child may elicit an automatic response for another child. For instance, the problem "How many stuffed animals do I have if I have six and three more?" is not a genuine problem to a child to knows how to represent this task by the expression "6 + 3" and knows immediately that 6 + 3 = 9. But for a child who does not yet know how to represent the task, or who has yet to learn that 6 + 3 = 9, this is not only a genuine problem, but one that can be used to develop the skills needed to solve other similar problems.

What does problem-solving look like in the pre-K–2 years?

Young children are constantly faced with new situations. In their normal explorations, they exhibit both intelligence and flexibility. This intelligence and flexibility is a great asset as problem situations arise naturally from school and everyday experiences. Problem situations are plentiful. How many paintbrushes do you need for your group? How many children are present today? How many are absent? Are there more boys than girls? Who has more blocks? How many more blocks would you need to have the same number as me? Which object is biggest? Are there enough juice boxes for everyone to have two? How many bites will it take you to eat that apple? How shall we divide the class into teams? How many more days until school vacation?

Problem posing—that is, generating new questions around a problem context—is part of the mathematical disposition that needs to be nurtured in the primary grades. "What happens if I have to share my pizza with 4 children instead of 3? If we are tossing pairs of number cubes, will we ever get a pair that adds up to one? What if we subtract instead of add? Then there would be lots of ways to get one."

Primary children will show great perseverance if they are interested in a problem. For example, one kindergartner wanted to know all the ways to make a hexagon using pattern blocks. At first he worked with the manipulatives using fairly undirected trial and error. As he gained experience, he became more methodical. And, once he had moved to a pattern block environment on the computer, his work became even more systematic until he could easily predict how he would have to finish a construction he had already started, and which attempts could not work.

Children working together tend to come up with many approaches to a problem. A group of children in one class was working to figure out how many pancakes Annie's mother has to make if Annie and seven friends are having breakfast and every child wants two pancakes. One child drew a square for each person, put two blocks in each, and counted. Another said said "I plussed it. 8 + 8 means 16." Another said, "I drew 8 sticks. Then I drew another 8 sticks and I counted." And one child, holding up one finger at a time until he came to eight, said, "2, 4, 6, 8, 10, 12, 14, 16. Sixteen pancakes." These children used a variety of strategies to solve the problem—physical representation, diagrams, and patterns.

Primary children use problem solving strategies to do problems that they do not yet have the knowledge to solve conventionally. For example, a first grader announced to her teacher that she knew how to multiply. "That's great," the teacher said. "Can you tell me 7 times 3?" The child knew that she had to add 7 + 7 + 7, but she had not yet committed this fact to quick recall. She did know, however, that 7 = 2 + 5. So she broke the problem up into easier pieces: "7 + 2 is 9. 9 + 5 is 14. That's two of the sevens. 14 + 5 is 19. 19 + 2 is 21. That's another seven. That's all three sevens. 7 times 3 is 21." This student also demonstrated important mathematical disposition—the disposition to believe that she could do it, and to stay with the problem.

What is the teacher's role in problem solving?

For children to engage in true problem solving, teachers need to identify and utilize situations that are both problematic and attainable for a wide range of students. They need to ask thought-provoking questions.

As students work on problems, they should be encouraged to share their thinking and approaches with other students and with adults. Sharing approaches to a problem is important for many reasons. It challenges children to articulate and make precise what they have done. It gives them opportunities to compare strategies. In the comparison, one way to solve the problem may seem easier because it requires less knowledge and another way may be more efficient. Two strategies that appear different may, upon closer inspection, be much the same. For example, the strategy "8 + 8 means 16" is the fluent version of drawing 8 sticks, 8 sticks more, and then counting. Through sharing approaches, children learn from each other. This is especially important as some children may have trouble with a particular problem. Seeing a variety of successful solutions modeled improves their chance to see a method they understand and can use.

Teachers should call attention to the strategies students and their classmates are using. In the pancake problem above, the teacher can help the children notice the strategies they used. "Caitlin, you used blocks." "Rebecca, you drew a picture." "Andrew, you used skip counting." Teachers can encourage monitoring and reflection by saying, "Tell me what's going on here. How did you make this list?"

Problems should be chosen by the teacher because they are likely to prompt particular strategies. For example, a problem that was introduced in the Overview of this draft document—"I have pennies, dimes, and nickels in my pocket. If I take three coins out of my pocket how much money could I have taken?"—can help children learn to think systematically and to carefully record their work. As students work on the problem, the teacher may want to focus on the way students organize and record the data.

Problems that help develop mathematical knowledge should be embedded into the rest of the mathematics curriculum, and are perceived by teachers or students as frivolous add-ons. The pancake problem above integrates ideas from arithmetic, number, connections, and representation. Problem solving can both lead to and confirm children's knowledge and understanding of mathematical concepts. For example, after an estimation activity in a kindergarten class, the children learned there were 83 marbles in the jar. There were 25 children in the class, so the teacher asked how many marbles each child could get. Graham said "3." The teacher asked, "How did you know that?" Graham replied, "83 is just a little more than 75, so we only get 3." "I am still confused," said the teacher, probing Graham's understanding. Graham went on to explain, "There are four quarters in $1. There are three quarters in 75¢. So we can only get 3," said Graham. "How many will be left over?" asked the teacher. Graham was able to transfer his knowledge from one context, money, to solve a problem involving discrete quantities and division. Thus, his knowledge base was confirmed, while his just-developing understanding of division deepened.

Problem solving is a process that permeates the mathematics program. In a classroom supportive of problem-solving, thought-provoking environments, questions, and investigations are the norm.

Standard 7: Reasoning and Proof

Mathematics instructional programs should focus on learning to reason and construct proofs as part of understanding mathematics so that all students—

- **recognize reasoning and proof as essential and powerful parts of mathematics;**
- **make and investigate mathematical conjectures;**
- **develop and evaluate mathematical arguments and proofs;**
- **select and use various types of reasoning and methods of proof as appropriate.**

Elaboration: Grades Pre-K–2

Young children strive to make sense of their world. They mentally incorporate everything that is happening to them into the mental structure they hold at the moment. This structure is in turn modified by their continuous experiences. Their experience with mathematics is no different. Children's early experiences with mathematical reasoning—their early attempts to make sense of mathematics—are crucial to their later mathematical development. Children need to be nurtured and encouraged so that they do not give up on the innate belief that the world, including mathematics, is supposed to make sense.

A major goal of mathematics instruction at pre-K–2 is to help children develop habits of thinking clearly and checking new ideas against what they already know. Young children will not demonstrate the chains of deduction that we often think of as mathematical reasoning, but they can and do make conjectures and conclusions that are logical and defensible from their perspective. Although young children very much want to make sense of the world, they have yet to develop all of the tools that adults use to make mathematical sense. For example, five markers spread far apart may be perceived as "more" than five markers close together. Maturity and experiences together help determine the development of reasoning.

Two elements central to the ability to reason are pattern recognition and classification skills. Young children naturally generalize from examples and it is important to help children judge whether generalizations they make are correct. The notion that examples can be used to test a conjecture is an important one, and by the end of second grade students should be using this method. The goal in pre-K–2 is to help children develop useful ways of making sense of mathematics and using these ways flexibly in new situations.

What does reasoning look like in pre-K–2 years?

Classification is a major focus in pre-K–2 mathematics, and it takes up much of the child's time from the pre-school years through second grade. Children's reasoning about their classifications varies. For instance, when a group of young children sorts shapes, one child may pick up a big triangle and say, "This one's big," and then put it with other large shapes. A friend may pick up another big triangle, trace its edges, and say, "Three sides—a triangle!" and then put it with other triangles. These children both are reasoning, but they are each focusing on only one attribute. By second grade, however, children are more aware that shapes have multiple properties and can suggest ways of classifying that will include multiple attributes.

Pre-K–2 students are developing the ability to reason systematically and carefully, and they should be given opportunities and time to do so. Patience and perseverance are important mathematical dispositions required for successful reasoning. Attribute games are activities that are accessible to children, as illustrated in this first grade class where students were playing the "Two Change Attribute Game." Most of the shapes in the set had already been placed in a straight line when it was Shanna's turn. She had to find a shape that was different in two ways from the previous shape, which was a large, thin, yellow circle. She took a long time to choose but finally said, "I found a large, thin blue rectangle. This piece is the same size and thickness, but it is a different color and shape. That's two changes! I wanted a little, thick yellow circle, but it was gone." The teacher then involved the class in a discussion about whether there were any other shapes that had not yet been placed that would have worked. During the discussion, she pointed out that Shanna had been able to change what she was looking for because she recognized what was the same and what was different about the shapes.

Creating and describing patterns offer important opportunities for children to learn to make valid arguments. The student who created the pattern shown in figure 4.10 proudly announced to her teacher that she had made four patterns in one. "Look," she said, "there's triangle, triangle, circle, circle, square, square. That's one pattern. Then there's small, large, small, large, small, large. That's the second pattern. Then there's thin, thick, thin, thick, thin, thick. That's the third pattern. The fourth pattern is blue, blue, red, red, yellow, yellow."

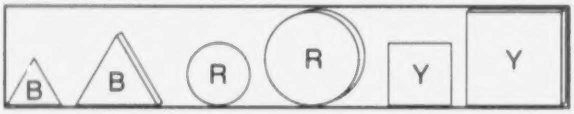

Figure 4.10. Four patterns in one.

Her friend, who was watching, studied the pattern for a minute and then said, "You could say that, but if you think about it another way, there are just two patterns. See, the shapes are in an AABBCC pattern. The sizes are an ABABAB pattern. The third pattern is also ABABAB. And your color pattern is same as the first, AABBCC. So, you really only have two different patterns." The first child carefully considered her friend's argument and then replied, " I guess you're right—but so am I!"

Finding patterns such as those on the hundreds chart encourages children to explore and make conjectures. When asked about all of the patterns they see on a hundreds chart, children will frequently describe patterns including the changes in numbers as you move down a column or across a row, patterns of number that appear in odd and even numbers, the diagonals, and many others. In one classroom, children were asked to color every third number. Different children saw different patterns, such as "Some rows have three and some have four" and "The pattern moves to the left." Some children, noting the diagonals in the pattern, no longer needed to count by threes in order to complete the pattern.

How do teachers facilitate the development of reasoning in pre-K–2 grades?

Teachers need to create learning environments that help children recognize that all mathematics can and should be understood and that they are expected to understand it. The pre-K–2 classroom needs to be well stocked with physical materials so that children will have many opportunities to manipulate concrete objects, to identify likenesses and differences, and to state generalizations. Within this environment, children will discover and demonstrate general mathematical truths using specific examples. For instance, a kindergarten student described to the teacher a discovery he made in the block corner. "I learned that a triangle equals a square," he announced. Puzzled, the teacher said, "Tell me more." The child demonstrated by going to the block corner and getting a half unit block and a triangular ramp (shown in figure 4.11). "You see," he said, "these are the same."

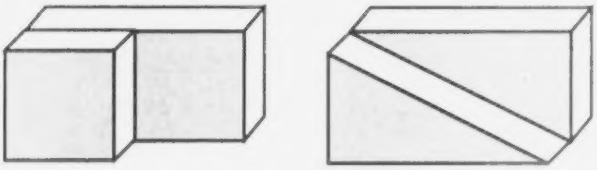

Figure 4.11. Equal blocks.

The child put the half unit block onto a unit block and said, "See, that's half." Then he put the triangular ramp on a unit block and said, "That's half and I can prove it." He got another triangular ramp to complete his covering of the unit block. "See," said the child, "they are the same, a square is a triangle." Even though the student's naming of the equivalent areas was not correct (a square is a triangle), the student demonstrated powerful reasoning and using the blocks to demonstrate his ideas.

From the very beginning, children need to have experiences that help them develop clear and precise thought processes. This development of logical reasoning is closely related to children's language development and is dependent upon children's abilities to explain reasoning rather than providing simple responses. As children learn language, they acquire basic logical words, including "not," "and," "or," "all," "some," "if…then," and "because." By using such words frequently, teachers can help children gain familiarity with the language of logic. For example, a teacher may say, "You may choose an apple or a banana for snack, but not both," or "If you hurry and put your jacket on, then you will have time to swing and play soccer." Later, children will use these words to describe mathematical situations, such as "If 6 green pattern blocks cover a yellow hexagon, then 3 blues also will cover it, because 2 greens cover blue."

Teachers can elicit children's thinking from children's answers. Upon hearing that he would be "Star of the Week" in half a week, Ben protested, "You can't have half a week."

When his teacher asked why, Ben said, "Seven can't go into equal parts." Ben had the idea the number seven could not be divided in half and he was reasoning from that premise.

Teachers should provide opportunities and encourage children to make conjectures and justify their thinking with reasonable arguments. They also should provide time for children to reflect so that they can develop logical arguments.

Standard 8: Communication

Mathematics instructional programs should use communication to foster understanding of mathematics so that all students—

- **organize and consolidate their mathematical thinking to communicate with others;**
- **express mathematical ideas coherently and clearly to peers, teachers, and others;**
- **extend their mathematical knowledge by considering the thinking and strategies of others;**
- **use the language of mathematics as a precise means of mathematical expression.**

Elaboration: Grades Pre-K–2

Communicating about mathematical ideas helps children to articulate, clarify, organize, and consolidate their thinking. It also helps them reflect on their own knowledge and strategies. Young children use many forms of communication—oral, pictorial, and symbolic. When children share what they know with other children and with the teacher, they both affirm and challenge their knowledge and also see it more clearly and coherently. Throughout the pre-K–2 years, children increasingly develop mathematical understanding as they communicate through representations, discussions, reading, writing, and listening. By listening carefully to other children and to the teacher, students learn alternative perspectives and strategies. Gradually, children learn the more precise language of mathematics and the use of conventional mathematical symbolism to express their mathematical ideas.

Discussion

What does communication look like during the pre-K–2 years?

Children begin to communicate mathematically very early in their lives. They want more milk, a different toy, and three sips from the water fountain. There is tremendous growth of communication abilities of children during the pre-K–2 years. This growth is determined to a large extent by the child's maturity, how language is modeled for the child, and his or her opportunities and experiences. Verbal interaction with parents and caregivers promotes the development of an appropriate mathematical vocabulary.

As children enter school, their opportunities to communicate are expanded by the new learning resources, enriched use of language, and experiences with classmates and the teacher. By comparing labels for different objects, for example, children expand their own ideas about classification. In early school years, children first use their own language to describe mathematical ideas. For example, they say "corners" and "points" rather than "angles" and "vertices." Gradually, teachers introduce the vocabulary and procedures of mathematics.

Young children have ways of communicating that are different from those used by adults. Children may use facial expressions, movements of their bodies, and objects in addition to oral language as they discuss and explain ideas. Manipulating objects and drawing pictures are as natural for young children as speaking. Later, they can use diagrams and can gradually be introduced to the use of mathematical symbols. By using language to relate physical materials, pictures, diagrams and symbols, children develop their mathematical ideas.

Children discuss mathematical ideas when they work in pairs or small groups on a common task. They share the results of their individual or small group findings with the teacher and the rest of the classroom. They need to listen attentively to each other, question the strategies and results, and ask for clarification.

How can teachers help pre-K–2 children communicate their mathematical ideas?

A major responsibility of pre-K–2 teachers is to help students learn to communicate mathematical ideas effectively by talking and listening to peers and teachers. Learning to analyze and reflect upon what is said by others is essential to seeing someone else's perspective. As children talk about their ideas, they reflect on their own thinking and clarify and strengthen their own understandings. Talk has the advantage of being flexible. It enables children to change directions easily, modify their first thoughts, express ideas in different ways, and even make new connections as they speak.

At a young age, a child's abilities to talk and listen are usually more advanced than his or her abilities to read and write. Students at this level should experience varied forms of communication as a natural component of mathematics class. Teachers should encourage students to use their own language. And, based on these expressions, teachers should model and help students build conventional mathematical vocabulary. Teachers also should encourage children to use a combination of pictures and diagrams in addition to short written text and symbols to communicate their thinking in mathematics.

Children should share their strategies and procedures as well as their solutions with teachers and peers. A first grader knew the answer to the problem, "How many books does the teacher need to return to the library, if she has 3 nonfiction and 4 fiction books?" When asked, the student responded quickly, "7." However, when the teacher asked him how he figured out the problem, he just shrugged his shoulders. Another student chimed in, "I do that this way." She held up three fingers and explained, "See, I know there are 3, so I just put up 4 more fingers and then count them." The first student listened and replied, "I did it a different way. I just knew that 3 and 3 make 6 and then I added 1 more!" The two students explained their strategies and all students had the opportunity to hear about and discuss the two ways of thinking about the same problem.

Adequate time and interesting materials, problems, and activities, including calculators and computer activities, encourage talk among small groups of children, much of it related to mathematics. In another example, Elle, who was usually a quiet child, delighted in teaching everyone in her first grade class how to count by ones and twos to 100 on the calculator. But, Elle became puzzled when counting to 100 by threes. "It always goes over 100!" she exclaimed. She and three friends decided to investigate this phenomenon. They talked together excitedly over the several days it took to complete the investigation and record the results on a chart, which they used to explain their investigation to the rest of the class.

In early primary years, the teacher will use communication in order to provide maximum success for all. For example, because not all children at this level are able to follow written instruction, the teacher might decide to read instructions. The issue of bridging the gap between allowing the students to read by themselves or be read to by the teacher is one that must be addressed case by case. It is important to remember that, in a mathematics lesson, ways of communicating should foster rather than impede the learning process.

Encouraging children to talk about mathematics lets teachers gain insight into those students' levels of knowledge and understanding in a given area. Correct answers sometimes hide misunderstandings about properties or procedures. Communicating mathematical knowledge and skill exposes what young children have internalized. Communication brings to the forefront children's mathematical thinking, so that the ideas can be analyzed and discussed. When children communicate, the teacher can assess the degree of understanding and progress of the children.

In addition to providing information about other skills, a benefit of pictorial or written statements is that they give the teacher information to share with parents and administrators about students' progress, conceptions and misconceptions.

What should students learn about communicating mathematically at grades pre-K–2?

Children learn to consider other points of view from hearing other children share different approaches and strategies. Through discussion, they gain new insights into other perspectives and views. When it is difficult for young learners to follow the reasoning of a classmate, the teacher can help by restating the reasoning provided by the child. Also, probing questions can guide students to rephrase their reasoning in words that are easier to understand for others.

In addition to talking and listening, pre-K–2 students need to write and read about mathematics. For example, after reading a story about a journey from a farm past a log cabin, over a hill to a school, across a bridge, and into the woods, a class of kindergarten children worked in cooperative groups to make a large map of the journey. As they drew, some children incorporated letters or other symbols into the work. As each group shared its work with the entire class, the students discussed the difficulties they had and what changes they would make in their work. The maps were hung in the hall, and children took the opportunity to analyze each other's work. Some said, "You make really good trees, but they should be a litter closer to the bridge." Others said, "We should have made our bridge a little more in the middle."

Providing activities that are engaging and relevant is likely to produce a higher level of mathematical conversation than repetitive assignments. Teachers who want children to communicate mathematically must provide something interesting for them to communicate about. At this age level, purposeful play is a particularly powerful form of activity that fosters mathematical engagements.

Invented algorithms allow children to manipulate numbers in ways meaningful to them and provide windows into students' reasoning, thinking, and level of understanding. Invented algorithms provide a natural link between informal and formal mathematics. Writing these algorithms helps children record and organize their thinking. Writing also helps students have a reference when they are explaining their ideas, helps them to see errors, and helps them prove the validity of their thinking.

It is the responsibility of the teacher to recognize appropriate times to bridge the gap between invented symbols and standard notation. The use of mathematical symbols should follow—not proceed—other ways of communicating mathematical ideas. In this way, teachers can help young children relate in a meaningful way their everyday language to mathematical language and symbols.

In summary, language is a very powerful tool and should be used to foster the learning of mathematics. Children can use both oral and written communication in mathematics as a way to solve problems, to articulate their thinking, and to reflect on their own mathematical understandings and those of others. It is essential that children develop facility with the specific language of mathematics in order to communicate their mathematical ideas effectively. By the end of this grade band, children should to be able to present ideas using their own and conventional notation and language. Learning new ways to represent mathematical ideas will continue to be a focus of instruction well beyond the primary grades.

Standard 9: Connections

Mathematics instructional programs should emphasize connections to foster understanding of mathematics so that all students—

- **recognize and use connections among different mathematical ideas;**
- **understand how mathematical ideas build on one another to produce a coherent whole;**
- **recognize, use, and learn about mathematics in contexts outside of mathematics.**

Elaboration: Grades Pre-K–2

Very young children do not view mathematics as a subject, apart from the rest of their world. The natural connections that young children develop need to be reinforced and continued. Adults who work with children need to develop an "eye" for mathematics in the world around them so they can find and spotlight the many situations in which mathematics is encountered and used by children both in and out of school. By expanding children's view of mathematics and showing how mathematical ideas are connected to each other, thinking and doing mathematics continues to be a natural and useful part of their everyday experiences. Children should come to value mathematics as they realize its usefulness. Making connections eliminates the barriers that separate the mathematics learned in school from the mathematics learned elsewhere. Making connections helps children realize that mathematics is a way to more clearly observe, represent, and interpret the world around them.

What do mathematical connections look like in the pre-K–2 years?

When a preschool child asks an adult, "Am I this many years old?" as he holds up three fingers, he is trying to make the connection between the word "three" and the number that represents his age, through a set of concrete objects, his fingers. Concrete objects and physical actions are important ways in which children connect mathematics to the world.

Young children often connect new mathematical ideas with old ones using concrete objects. The problem of making all combinations of five using two colors of connecting cubes helps children understand addition, but also involves pattern and connects the concept of number with geometry and measure. As children discover that they can make five combinations with four cubes of two colors and six combinations for five cubes, they are likely to predict that there will be seven combinations for six cubes using two colors.

Mathematics is embedded in the entire curriculum. For example, in physical education class children count the number of times they can jump rope successfully. They measure and compare the amount of time each team member takes to move from one end of the gym to the other while doing sprints. Geometric shapes are reinforced as children form a circle to play a game, take their place around the perimeter of the parachute, or assume their positions on the softball diamond. Students explore symmetry in art class, they classify in science class, and they sing songs rich in patterns in music class. In one classroom, the teacher clapped out the rhythm of a child's name (one clap per syllable) and asked the children if they could figure out whose name she was clapping. When her rhythm matched the names of several children, the group began to try to find other children who had the same number of claps in their names. They drew the chart shown in figure 4.12.

3 beats	**4 beats**	**5 beats**	**6 beats**
John Gosha	Sarah Andrews	Bobby Erickson	Sylvia McPherson
Timmy Simms	Carlos Sanchez	Anisha Johnson	Alyssa Huerrero

Figure 4.12. Beat names.

When one child could not find anyone with seven beats, another child illustrated seven beats by including the beats in her middle name in her total. Other children then began to experiment with middle names and nicknames to match each other's names. One child looked at the chart and questioned whether John Gosha and Timmy Simms were actually "beat twins." Even though the number of beats was the same, the names sounded different. The students began to record their name beat patterns in a more specific way.

John Gosha $1 + 2 = 3$ Timmy Simms $2 + 1 = 3$

In this example, children used mathematics to investigate and quantify rhythm. As the students determined the best ways to describe rhythm through number, they also were reinforcing their arithmetic skills. Such interplay, in which mathematics illuminates a situation and the situation illuminates mathematics, is an important aspect of mathematical connections.

What is the teacher's role in helping students make connections in pre-K–2?

Teachers must ensure that links to mathematics continue in routine school activities, including measuring water to carry to the painting area, measuring paper to make a headband, sharing crayons fairly, and being in charge of the milk and lunch count. In classrooms where connecting mathematical ideas is a focus and goal, lessons are fluid and take many different formats. Teachers revisit topics previously taught to establish new links between mathematical concepts and procedures. Teachers capitalize on unexpected learning opportunities, such as the music lesson described above. Teachers provide extended projects and investigations in which students can make important connections through their own experiences as they formulate questions, design inquiries, make interpretations, decide on methods of recording information, and create meaningful representations. In these ways, children become more able to think independently of the teacher and assume responsibility for their own learning.

Many mathematical situations for the young child are ambiguous, and teachers can use the way a child interprets these situations as clues to that child's understanding. A student

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who asks the teacher to name a pattern block shape, a student who builds a pattern block design and directs the teacher's attention to its symmetry, and a student who tells the teacher that the pattern block trapezoid is half of a hexagon are all seeing different things in the same set of objects. Thoughtful teachers listen to children to assess the knowledge and skills children bring to their situation, and use this information to plan activities that will further children's mathematical knowledge and skills.

Young children enter school with a natural tendency to connect everything they know. It is the responsibility of the teacher to build on this tendency to help children see and experience the interrelation of mathematical topics, the relationships between mathematics and other subjects, and the way that mathematics is embedded in the child's world. Establishing connections is a focus of teaching and learning that helps students develop a strong understanding of mathematics and the mathematical skills to further their learning.

Standard 10: Representation

Mathematics instructional programs should emphasize mathematical representations to foster understanding of mathematics so that all students—

- **create and use representations to organize, record, and communicate mathematical ideas;**
- **develop a repertoire of mathematical representations that can be used purposefully, flexibly, and appropriately;**
- **use representations to model and interpret physical, social, and mathematical phenomena.**

Elaboration: Grades Pre-K–2

Through mathematical representations children express their thinking about, understanding of, and use of mathematics. These same representations give insight into a child's grasp of mathematical concepts. Primary children use oral language, idiosyncratic symbols, dramatizations, and drawings to represent mathematical ideas. These representations are used by children both in understanding mathematical ideas and in showing the results of their work. By talking about mathematics young children can better understand mathematical relationships and link ideas. Representations provide a way for the teacher to access children's understanding and see how they think about mathematical ideas.

What does representation look like in the pre-K–2 classroom?

Young children represent their mathematical ideas and procedures in many ways. They can use manipulative objects, their own fingers, natural language, drawings, diagrams, and dramatizations. Through interactions with these external representations, children develop their own mental images of mathematical ideas. Although the representations that children use may not be the same as the ones traditionally used by adults, children's representations provide a record of their efforts to understand mathematics and at the same time make their understanding available to others.

Children naturally represent their mathematical ideas using their own methods. Their representations are often insightful and many times resemble more formal representations. For example, a second grader working with place value mats and base-ten blocks has represented "103." She points at the blocks and taps at the empty column as she reads "one hundred, (tap), and three." The tap helps the child connect the zero with the empty tens column.

Children use representations to make mathematical ideas more concrete and manageable. They can represent ideas with objects that can be moved and rearranged. Such concrete representations provide the foundation for later representations with pictures and symbols. Children can also use representations to organize their thinking. For example, a child trying to find the number of wheels in four bicycles and three tricycles drew the following diagram (figure 4.13).

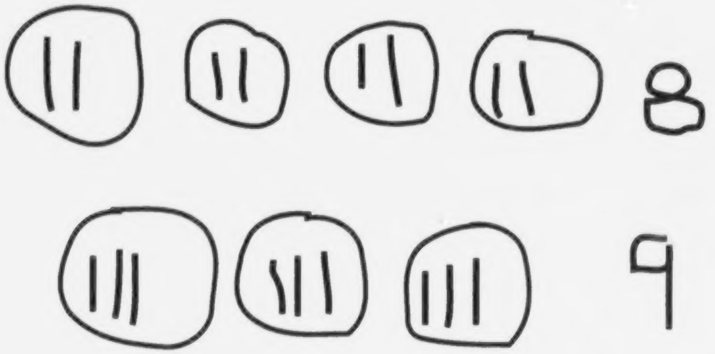

Figure 4.13. Wheels on bicycles and tricycles.

By representing the number of wheels in the tricycles with one set and the number of wheels of the bicycles with another set, the child was able to compute the sums separately and add them together. Representations can carry some of the burden of thinking by letting children record intermediate steps and carry out some of the steps of the process without having to worry about the rest for the moment. The representations can serve as placeholders for thoughts that are not yet internalized.

Understanding and use of mathematical concepts and procedures is enhanced when children can represent mathematical ideas in multiple ways and when they can translate between different representations of the same idea. For example, a child who represents four groups of four squares as an array and uses skip counting (4, 8, 12), repeated addition (4 + 4 + 4), and oral language to describe the representation as three rows of four squares or as four columns of three squares is laying the groundwork for understanding multiplication and its properties, as well as the area of a rectangle. Children come to appreciate that some representations highlight in a better way some of the features of the problem or make it easier to understand certain properties.

How can teachers use representations to facilitate students' understanding?

Teachers should provide as many different representations as necessary for children to reach an understanding of mathematical concepts and procedures. Representations can help students recognize the common mathematical nature of different situations. For example, in one classroom, first graders are dealing with several problems. The first problem involves taking away three objects from a collection of five. The second problem is about how much taller a tower of five cubes is than a tower of three cubes. The third problem is about how many more balls they need to put in a box, if they want to have five and there are already three in the box. The subtraction sentence 5 - 3 = 2 represents all of these situations. This is an important step toward abstraction.

Children's representations reflect their thinking. The transition to more conventional notations should reflect the methods and thinking of the children. For example, when children use blocks or mental computation to solve a problem like "17 + 25," they commonly add the tens first. Teachers can help children use symbols to record their intermediate steps as 10 + 20 = 30, 7 + 5 = 12, and 30 + 12 = 42. Later, with guidance from the teacher, children also can write this process in vertical notation, where the first partial result of adding 10 + 20 is written immediately below the horizontal line:

$$\begin{array}{r} 17 \\ \underline{+\,25} \\ 30 \\ \underline{+\,12} \\ 42 \end{array}$$

Representations provide a record of children's thinking that shows both the answer and the process. Using representations makes it is easier for children to remember what they did and to explain and justify their reasoning. These communications serve both students and teachers. They inform the questions that teachers must ask to help students reflect on their processes and products. These representations also can be used for assessment or to guide instructional planning.

It is important that students and teachers realize that any representation, not only those created by children, are subject to multiple interpretations. Drawings, charts, graphs, and diagrams, for instance, can be read in different ways. Therefore, it is important not to assume that children understand a diagram or equation the same way adults do. It is the pre-K–2 teachers' responsibility to help children develop proficiency with the use of multiple representations. In so doing, children gain better understandings and insights. Representing the same problem using alternative representations may bring additional comprehension by emphasizing different aspects of the situation. On the other hand, using the same representation for several situations may highlight the common underlying mathematics in them.

Teachers of primary children need to listen to what students say, thoughtfully observe their mathematical activities, analyze their recordings, and reflect upon the implications of each. The data gathered from these multiple sources give a clearer picture of what children understand and what mathematical ideas a still developing. With this information, teachers can plan and effectively use representations as tools to model mathematical phenomena.

Chapter 5
Standards for Grades 3–5

Mathematics is a subject most students in grades 3–5 like. In fact, nearly three quarters of fourth graders report liking mathematics (Kenney and Silver 1997). Students in grades 3–5 see mathematics as practical, are challenged with many new ideas, and believe that what they are learning is important. However, sometime between grades 4 and 8 students' interest in mathematics begins to wane. Although they continue to view mathematics as important, by grade 8 students are less likely to characterize it as interesting or to consider themselves good at mathematics (Kenney and Silver 1997). It is crucial that the mathematics education in the upper elementary and early middle grades be challenging, relevant, and engaging for students.

The curriculum materials and instructional approaches a teacher uses help students connect mathematical ideas and provide a basis for making them meaningful. Because the amount of content in grades 3–5 expands greatly from that of the earlier grades, students need help in building connections and managing the many new concepts and procedures they are encountering. Students in grades 3–5 must also understand and bear responsibility for their learning. In mathematics this means learning how to examine, ask questions, and consider different strategies—all with the goal of making sense of mathematical ideas and fitting them to other, related ideas.

The ten standards outlined in this chapter are common to all the grade bands, pre-K–12, and each of them should be included in a robust grade 3–5 mathematics program. The instructional emphases for this grade band build on those outlined for grades pre-K–2 and extend well beyond them. Although number and operation continue to be cornerstones of the curriculum in grades 3–5, each of the content standards (number, data, measurement, algebra, and geometry) is essential for building student knowledge at this level. Several "big ideas" or central mathematical themes for this grade band are woven through these areas, including multiplicative reasoning, equivalence, and the notion of unit. Likewise, knowledge and use of mathematical processes should be deepened and expanded in these grades. Students in grades 3–5 are capable of sophisticated reasoning and should be challenged and supported in their learning.

Content Issues

A major goal within the Number and Operations standard at this grade band is the development of computational fluency. *Fluency* refers to having efficient and accurate methods for computing. In some cases, these methods are mental strategies. In other cases they are paper-and-pencil strategies (either student-invented or conventional algorithms). Fluency depends on the efficient recall of basic facts and on well-understood ideas about number and operations. Students should have opportunities to develop their own strategies for computation that are based on their understanding of number properties and on the particular context of the problem. Over time, the teacher should help students develop efficient methods that they can keep track of and record clearly. Students may need help with refining and consolidating their own methods as well as with recording their work in an organized way. When a student's approach is consistently cumbersome, inefficient, or particular to the numbers in a given problem, the student should be helped to use and understand more efficient and general methods used by others, including student-invented and conventional computational algorithms. Students should come to view algorithms as tools for solving problems rather than as the goal of mathematics study.

In grades 3–5, extending understanding from whole numbers to fractions and decimals is a key dimension of the mathematics curriculum. Students need many and varied experiences in order to understand what fractions and decimals represent, how they are related to each other, and how they are different from whole numbers. These experiences should include drawing pictures of rational numbers, locating them on a number line, relating symbolic and physical representations (e.g., circles, number lines), comparing and ordering fractions using benchmarks such as 1/2, estimating sums of fractions (e.g., is the sum more or less than 1?), and developing an understanding of equivalent forms of rational numbers. Estimation and mental computation with common fractions can and should be encouraged, building on number relationships and visual models. The instructional time devoted to rational numbers in grades 3–5 should be spent building up conceptual understanding of fractions—what they are, how they are represented, and how they are related to whole numbers. These kinds of experiences will provide a good foundation for later work with rational number computation in grades 6–8. Fluency with the computation of fractions and the use of conventional algorithms for computing with fractions should be delayed until the middle grades.

Balancing Attention to Conceptual Understanding and Procedures

Students understand a mathematical idea when they can view it from multiple perspectives, relate it to other concepts and experiences, and use it as a stepping stone to think about more complex ideas. The new mathematical idea must also be usable in that the learner is able to apply it to solve a problem. Every new understanding builds a stronger foundation and allows a certain independence of thinking. Understanding mathematical ideas can stimulate and support the acquisition of skills as well as the ability to solve problems (Hiebert et al. 1997). Skills can be learned without understanding, but they are likely to be forgotten or misapplied. Likewise, understanding an idea without developing useful skills for applying it to solve problems is also not sufficient. Teaching for understanding is essential but also difficult; judging whether a student understands is more difficult than judging whether he or she can perform a procedure. Each student acquires understanding in different ways and at different times. Although it is a challenge, teachers must organize their instruction so that understanding provides a foundation for doing.

Mathematical procedures are important tools that allow students to apply understandings and solve problems efficiently. They are useful at every level of instruction and include such things as computational algorithms, formulas, techniques for solving equations, and methods for constructing graphs and tables. As students develop procedures, part of the teacher's role is to evaluate their work, to help them recognize and develop efficient procedures, and to support them in practicing those procedures so that they become fluent.

Teachers need to balance the dual goals of ensuring conceptual understanding and developing procedural knowledge. They must continually monitor student progress and make judgements about what to emphasize. At times, teachers emphasize concepts and encourage students to develop their own procedures based on their understanding. At other times, they help students recognize and appreciate the power of some conventional procedures that are elegant in their efficiency.

Technology as a Tool for Learning and Doing Mathematics

The calculator and a variety of computer software should be considered legitimate tools for learning and doing mathematics and should be available to students in grades 3–5. Teachers should create opportunities and make judgments about when and how these tools are used to support learning. As tools for doing mathematics, students should use calculators and spreadsheets to carry out the procedures needed to solve problems. As a tool for learning mathematics, students in grades 3–5 should use graphing and dynamic geometry software and calculators to explore, experiment, verify, and visualize mathematical ideas. For example, students in grades 3–5 should explore geometric relationships using software to create, modify, and examine shapes. They should create graphs and consider how some presentations of data highlight or distort certain trends. They should use calculators to explore the relationship of decimals to whole numbers or of negative numbers to positive numbers. They should also have the opportunity and learn techniques to search the Internet to gather information and solve mathematical problems.

The mathematics curriculum outlined in this set of standards reinforces the dual goals that mathematics learning should be both about making sense of mathematical ideas and about acquiring skills and insights to solve problems. The calculator is an important tool in reaching these goals in grades 3–5. However, calculators will not replace the need for quick recall of basic facts, conceptual understanding, or the ability to formulate and use strategies for computing. Rather, the calculator should support these goals by enhancing and stimulating student learning. As students solve problems involving many complex computations, the calculator should be used to perform calculations. The calculator serves as a tool for enabling the problem solver to focus on the big picture rather than become entangled in the calculational details. Calculators also can help students understand ideas. For example, using the calculator to skip-count by tenths or hundredths showcases patterns in decimals that are not evident through other models.

Need for Mathematics Teacher Expertise and Leadership

A critical issue at grades 3–5 is the development of teacher expertise. This expertise includes a thorough understanding both of mathematical content and of student's mathematical thinking in these grades. More and more, schools and systems that are serious about improving mathematics teaching and learning are turning to teachers themselves to help provide professional development and support within their own schools. The teachers most experienced in mathematics are given special assignments to support other teachers—sometimes these teachers are released completely from the classroom for a year or two; sometimes they combine half-time classroom teaching with half-time work in a mathematics support role;

sometimes they take on additional duties beyond their full-time classroom teaching. Given the scope of the task of improving mathematics in the elementary grades, it is essential to involve selected teachers in this enterprise. However, in order for teachers to become leaders, they must be given considerable support and must have the time and resources to do effective work with other teachers.

Some school districts are using a "mathematics specialist" to provide greater expertise at the upper elementary level for teaching mathematics. For example, mathematics instruction for a team of fifth graders in a particular school might be provided by one teacher, with another teacher to organize science instruction and yet another who is responsible for social studies instruction. This strategy allows individual teachers opportunities to focus their own professional development and planning time. Various mathematics specialist models exist, and each may present advantages and disadvantages, given a particular context. As the mathematics students are encountering continues to grow more sophisticated, it is critical that strategies be investigated to insure that their teachers have the needed mathematical and pedagogical expertise.

The remainder of this chapter elaborates the ten standards for grades 3–5. The concepts, procedures, properties, and relationships described in this chapter form the basis of a mathematics curriculum students in grades 3–5 can and should learn with understanding. Both the teacher and a comprehensive, coherent mathematics curriculum serve as a "loom" to help students weave facts and concepts into the fabric of mathematical knowledge.

#7

Reader Reaction

As you read the next sections, please consider the following:

- **Do you think a clear position is taken on the use of calculators in grades 3–5? Is enough guidance given? Please explain.**
- **How might we improve the discussion on the relative emphasis on student-invented and conventional computational algorithms?**
- **Do the sub-bullets for Standard 1, Number and Operation, provide enough guidance? Please explain and give specific suggestions.**

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Standard 1: Number and Operations

Mathematics instructional programs should foster the development of number and operation sense so that all students—

- **understand numbers, ways of representing numbers, relationships among numbers, and number systems;**
- **understand the meaning of operations and how they relate to each other;**
- **use computational tools and strategies fluently and estimate appropriately.**

Elaboration: Grades 3–5

In grades pre-K–2, students develop knowledge and skills about the order of numbers and the structure of the base ten number system and learn how numbers are related to each other in this system, with a major focus on tens and ones. They encounter many addition and subtraction situations and learn how to select and use operations to solve such problems. They develop computational strategies for the addition and subtraction of whole numbers. They begin to use and name common fractions in contexts such as making equal shares from a whole. Throughout this work, the focus has been on developing meaning for the operations, making sense of the size and relationships of numbers , and learning how to record and explain their work using concrete models, pictures, words, and numbers.

Building on this work, students in grade 3–5 turn to a focus on multiplication and division of whole numbers. Students enrich their sense of number as they develop a stronger understanding of the various meanings of these operations, encounter a wider range of representations and problem situations that embody them, learn about the properties of these operations, and gradually develop fluency in solving multiplication and division problems. Students also focus on extending their understanding of the base ten number system as they work with larger numbers—thousands and beyond—as well as decimals. They learn the meaning of fractions, how fractions are related to each other and to the unit, and gain facility in comparing fractions, often by using benchmarks such as 1/2 or 1.

When students leave grade 5 they should be able to choose operations appropriately to solve problems and recognize that each type of operation will help them solve many different problems. They should be able to estimate a reasonable result for a problem, solve many problems mentally, and compute fluently with whole numbers. They should view numbers as meaningful entities, understand how the number system grows from tens to hundreds to thousands and beyond, have quick recall of basic facts for all operations, and have a repertoire of strategies which they use to solve number problems. They should understand how fractions, decimals, and percents are written and modeled, how they relate to the unit whole and to each other. These understandings and skills contribute to the growth of number sense, reflected in persistence in seeking meaning and reasonableness in the results of mathematical work. With skills, knowledge, and a view of mathematics as making sense, students will be well prepared for solving numerical problems in many contexts and for a focus on rational numbers and proportional reasoning in the middle grades.

Focus Areas for Grades 3–5

- **Understand numbers, ways of representing numbers, relationships among numbers, and number systems**

  In grades 3–5, all students should—

  - understand the structure of the base ten number system including representations of decimals;
  - continue to develop a sense of the relative magnitude of numbers, with a focus on powers of 10 (especially hundreds and thousands) and their role as benchmarks in the number system;
  - read and write large numbers and relate notation to the meaning of these numbers;
  - develop meaning for fractions as parts of a unit whole, as a part of a collection, as numbers, and as a division of whole numbers;
  - read and write fractions and decimals and relate the notation to the meaning of these numbers;
  - develop strategies for judging the size of fractions and decimals and for comparing them using a variety of models and of benchmarks (such as 1/2 or .5);
  - recognize and use common fraction, decimal, and percent equivalents;
  - identify characteristics of particular numbers (e.g., is it odd or even? is it prime or composite?) and use this information to describe characteristics of classes of number (e.g., multiples of seven);
  - develop meaning for factors and multiples, become fluent with factor and multiple relationships, and classify numbers according to their factors (prime, composite).

- **Understand the meaning of operations and how they relate to each other**

  In grades 3–5, all students should—

  - understand the meaning of multiplication and division including multiple representations (e.g., multiplication as repeated addition or as an array);
  - identify and use relationships between operations to solve problems (e.g. multiplication as the inverse of division);
  - identify and use properties of operations to solve problems (e.g., 28 x 7 is equivalent to (7 x 20) + (7 x 8) or (7 x 30) – (7 x 2)).

- **Use computational tools and strategies fluently and estimate appropriately**

  In grades 3–5, all students should:

  - develop fluency with single-digit multiplication facts and their related division facts by grade 4 and use these facts to efficiently compute related problems (e.g., 30 x 50 is related to 3 x 5, 300 x 5, and 15 x 100);

- develop and compare whole-number computational algorithms for each operation that are based on understanding number relationships, the base ten number system, and the properties of these operations, and by the end of grade 5, develop efficiency and accuracy in using these algorithms;
- develop and use computational estimation strategies based on an understanding of number concepts, properties and relationships;
- estimate sums and differences of common fractions and decimals using benchmarks (e.g., 3/8 + 1/3 must be less than 1 since both fractions are less than a 1/2);
- develop and use visual models, benchmarks, and equivalents to add and subtract with common fractions (e.g. 1/2 + 1/4 is the same as 1/4 + 1/4 + 1/4 or 3/4; 1 – 1/5 leaves 4 fifths of the "pie" remaining);
- develop and use visual models, benchmarks, and equivalents to develop computational procedures for the addition and subtraction of decimals (e. g., determining 0.10 + 0.15 by shading in a 10 x 10 grid, calculating 0.75 – 0.25 by relating these numbers to fraction equivalents);
- choose appropriate computational procedures and tools (e.g., calculators, pencil and paper, mental computation, estimation) to solve problems.

Discussion

• Number Concepts and Properties

Students in grades 3–5 develop strategies for judging the relative size of numbers and use important benchmarks in the base ten number system, especially multiples of 10, 100, and 1000. It is not enough for students to be able to simply say that the number 886 has "8 hundreds, 8 tens, and 6 ones." They must know about the position of this number in the base ten number system and its relationship to benchmarks such as 500, 900, and 1,000. They need to learn about the effects of operating on numbers, such as adding or subtracting 10 or 100, or multiplying by 10. In order to do this, students should encounter these numbers in a variety of contexts in which they explore how the numbers are structured. For example, a third-grade class might explore the size of 1,000 by counting and writing numbers to 1,000, by constructing 1,000 using ten hundreds charts, by gathering 1,000 items such as paper clips and developing efficient ways to count them, or by using strips that are 10 or 100 cm long to estimate the length of 1,000 cm.

At these grade levels, students are also developing their knowledge of rational numbers. In grades 3–5, students should use a variety of models of fractions in order to expand their knowledge about fractions as numbers. By using an area model, students can see how fractions are related to a unit whole, compare fractional parts of a whole, and find equivalent fractions. By using parallel number lines, each showing a unit fraction and its multiples (figure 5.1), students can see fractions as numbers related to 1, and can see relationships among fractions.

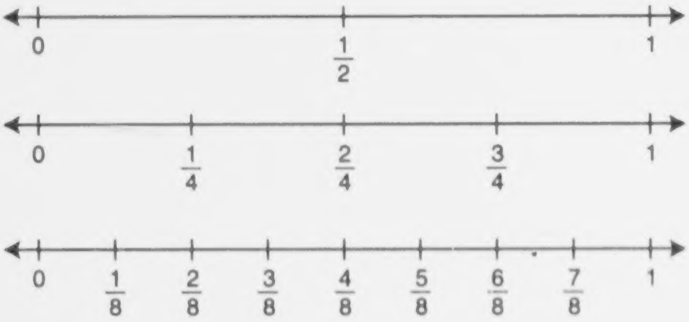

Figure 5.1. Fractions on a number line.

Students in grades 3–5 work primarily with familiar fractions, such as halves, thirds, fourths, sixths, eighths, and tenths, and focus on ordering and comparing fractions, often using benchmarks such as 1/2 and 1. For example, fifth graders can use their understanding of 1/2 to make comparisons of other fractions such as 2/5 and 5/8. They should also focus on finding equivalents of familiar fractions and on the meaning of the numerator and denominator of a fraction.

Students in grades 3–5 also represent, name, write, read, and compare decimals and percents. Decimals, like fractions, can be represented with an area model as parts of a whole or on a number line, showing their position as numbers. Features of the base ten notation can help students understand decimal notation. Just as moving to the left in the places of written numbers indicates an increase by a factor of 10, moving to the right—say, from ones to tenths or from tenths to hundredths—indicates division by 10.

As students relate fractions to decimals, they also encounter another meaning of a fraction as the division of two whole numbers (5/2 = 5 ÷ 2 = 2 1/2). By using the calculator to carry out the indicated division of fractions like 1/2, 1/3, 1/4, 1/5, 1/6, and so forth, they learn common fraction-decimal equivalents. They also learn that some divisions result in repeating decimals. By studying fractions, decimals, and percents simultaneously, students can use what they know about equivalence to solve problems. For example, a student might add 0.5 and 0.25 by using their equivalents, 1/2 and 1/4.

- **Operations and their Properties**

In grades 3–5, students focus on the meaning of multiplication and division and their relationship. It is critical that early in these grades students understand the meaning of the various quantities in a multiplication problem. In multiplication, unlike addition, the numbers in the problem can refer to different units. For example, if students are solving the problem 29 x 4 to find out how many legs are on 29 cats, 29 is the number of cats (or number of groups) and 4 is the number of legs on each cat (or number of items in each group). Through representing multiplication problems with pictures, diagrams, or concrete materials, students learn to be clear about what each number in the problem represents.

Further meaning for multiplication is developed as students build and describe arrays, showing how a product is related to its factors. This model is an important one to help students develop an understanding of how multiplication behaves (for example, that 3 x 4 = 4 x 3). Arrays also provide a model in which students can see relationships between factor pairs. For example, an array for 20 x 6 can be split in half and the halves can be rearranged to form an array of 10 x 12, showing that 10 x 12 = 20 x 6. By studying factors and multiples, students expand their knowledge of the properties of numbers (e.g. finding that some numbers are prime and others composite) and develop number sense about what happens when certain numbers are multiplied or divided (e.g., can a multiple of 20 always be divided evenly by 4?).

Division has different meanings, depending on the context. For example, if there are 116 people traveling by bus and each bus can hold 29 people, how many buses are needed? In this case, 116 ÷ 29 means that there are 116 people, that there will be 29 people in each group (bus), and we need to find the number of groups (buses). In a different problem, we might know the number of groups and need to find how many items are in each group. If 116 people divided themselves evenly onto four buses, how many people were on each bus? In this case, 116 ÷ 4 indicates that we know the total number and the number of groups (buses), but not how many are in each group. Students need to recognize both types of problems as division situations, should be able to model and solve each type of problem, and should know the units of the result—is it 29 buses or 29 people per bus?

In third grade, many students are likely to think of multiplication as repeated addition and division as repeated subtraction. As they work with a variety of problems and models in these grades, students should expand their views of multiplication and division. In particular, they should develop ideas about the distributive property of multiplication, on which many efficient solution strategies, including the standard multiplication algorithm, are based. For example, students might compute 7 x 28 by breaking up this problem in any of the following ways—all applications of the distributive property (figure 5.2.)

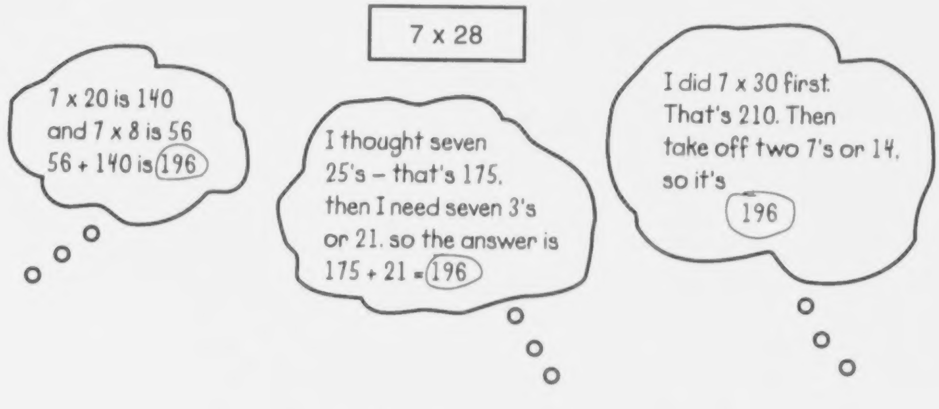
7 x 28
7 x 20 is 140 and 7 x 8 is 56 56 + 140 is 196
I thought seven 25's – that's 175. then I need seven 3's or 21. so the answer is 175 + 21 = 196
I did 7 x 30 first. That's 210. Then take off two 7's or 14. so it's 196

Figure 5.2. Solutions for 7 x 28.

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- **Computation and Estimation**

Students in grades 3–5 should acquire fluency in multiplying and dividing with whole numbers and should begin to extend their understanding of these operations to work with familiar fractions, decimals, and percents. The goal is for students to develop fluency in computing on the basis of their understanding of the base ten number system, their knowledge of a large repertoire of number relationships including the basic facts, and an understanding of the properties of the operations.

Early in grades 3–5, students begin to focus on learning multiplication and division relationships and thinking strategies that will assist them in their acquisition of the single-digit multiplication facts and their related division facts. In these grades, students should develop approaches to doing calculations for each operation based on mathematical ideas and relationships that have meaning for them. Important strategies include breaking apart the numbers in the problem, as in the first solution in figure 5.2, or creating an equivalent problem that is more accessible, as in using 7 x 30 as the starting point. Students using approaches like these are calling on their number and operation sense (i.e., using the properties of distributivity or associativity).

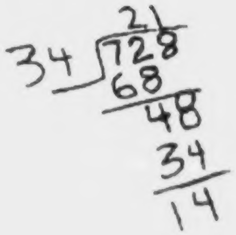
As students share strategies in class, they learn from each other, analyze the efficiency and generalizability of various approaches, and are encouraged to try one another's strategies. In the past, common school practice has been to teach a single algorithm for each operation. However, more than one efficient, accurate, and mathematically sound algorithm exists for each operation. Further, some methods of notation for algorithms tend to obscure, rather than illuminate, the meaning of the numbers and the steps of the procedure. By comparing and analyzing various approaches, students can learn more about the operation they are performing. The conventional computational algorithms for multiplication and division can be considered towards the end of grades 3–5 as one efficient way to calculate. As with all other approaches, students who use these algorithms should be able to explain and justify their procedures. The following example illustrates how one teacher helps students analyze and compare their computational procedures for division.

> Students in Ms. Sparks' fifth-grade class are sharing their homework: solving the problem 728 ÷ 34. Ms. Sparks asks several students to put their work on the board to be discussed. She deliberately chooses students who have approached the problem in several different ways. As the students put their work on the board, Ms. Sparks circulates among the other students, checking their homework.
>
> Henry has written his solution:

34 x 10 =	340
34 x 20 =	680
680	728
+ 34	– 714
714	14

> Henry explains to the class, "Twenty 34's plus one more is 21. I knew I was pretty close. I didn't think I could add any more 34's, so I subtracted 714 from 728 and got 14. Then I had 21 remainder 14."

Michaela has shown her solution:



Michaela says, "34 goes into 72 two times and that's 68. You gotta minus that, bring down the 8, then 34 goes into 48 one time."

Ricky: I don't know how to do that.

Michaela: You divide, then you multiply, you subtract, then you bring down.

Ricky: I still don't get it.

Ms. Sparks: Does anyone see any parts of Michaela's and Henry's work that are similar?

Christy: They both did 728 divided by 34.

Ms. Sparks: Right, they both did the same problem. Do you see any parts of the ways they solved the problem that look similar?

Fanshen (hesitantly): Well, there's a 680 in Henry's and a 68 in Michaela's.

Ms. Sparks: So, what is that 68, Michaela?

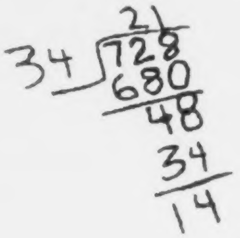
Michaela: Um, it's the 2 x 34.

Ms. Sparks: Oh, is that 2 x 34?

Samir: Oh! It's 20 times 34. It's like the 20 x 34 that Henry did.

Henry: But 20 times 34 is 680, not 68.

Ms. Sparks: So what if I wrote a 0 here to show that this is 680? Does that help you see any more similarities?



Maya: They both did twenty 34's first.

Rita: I get it. Then Michaela did, like, how many more are left, and it was 48, and then she could do one more 34.

As the class continues their study of division, Ms. Sparks will continue to return to this comparison of computation strategies in order to help all students explain and justify their approaches.

As students move from third to fifth grade, they should consolidate and practice a small number of strategies for multiplication and division that they understand well and can use them fluently. Developing more than one approach for each operation allows students to choose an approach that best fits the numbers in a particular problem. For example, to solve 298 x 42, it may be easiest to think of the problem as (300 x 42) – (2 x 42), whereas 525 x 12 is easily solved by multiplying 525 first by 10, then by 2, and adding the two subproducts.

Students should have ample opportunity to use mental computation and should learn to choose carefully among mental approaches, use of paper and pencil, use of the calculator, or combinations of these approaches. During these grades, students will frequently use calculators to solve problems. However, there will also be times when they are honing their skills with basic facts or mental and written computation without calculators. The goal in grades 3–5 is for students to calculate accurately and efficiently with whole numbers of reasonable magnitude—numbers they are likely to use in daily life—using strategies that make sense to them and, at times, tools such as calculators. In addition, they should be able to calculate with larger numbers that have special characteristics in our number system (e.g., what is 100 times 100? what is 10,000 plus 2,500?).

Students will develop computational estimation strategies at these levels and begin to regularly assess the reasonableness of problems and solutions. For example, before beginning to solve a problem, they should look at the numbers in the problem and think about how carrying out the operation will affect those numbers. Will the answer be larger than 50? Larger than 100? Close to 1000? More than 1000? What upper and lower boundaries can be set for the solution (e.g., 68 x 34 must be between 1800 and 2800, since 60 x 30 is smaller than 68 x 34 and 70 x 40 is larger). In addition, students should encounter situations in which estimation is the process of choice. When an estimate is what is needed, the student must consider how close an estimate is good enough and whether it is better in the particular context for the estimate to be too low or too high.

With the grounding students have developed in the meaning and relationships of fractions and decimals, they can begin to solve a variety of problems involving familiar fractions and decimals. At these grades, the emphasis is not on developing general procedures to solve all fraction problems but on developing solutions based on students' sense of number and using area and number line models. For example, in a fifth-grade class, students worked on this problem: Jamal invited 7 of his friends to lunch on Saturday. He thinks that each of the 8 people will eat 1 1/2 sandwiches. How many sandwiches should he make? Students might draw a picture and count up the number of sandwiches or might use reasoning based on their knowledge of number and operations: That would be 8 whole sandwiches and 8 half sandwiches; since two halves make a whole sandwich, the 8 halves will make 4 more sandwiches, so Jamal needs to make 12 sandwiches. Students should also use their knowledge of fraction relationships to estimate a reasonable result (e.g., 1/3 + 2/5 must be less than 1 since both fractions are less than one half).

Standard 2: Patterns, Functions, and Algebra

Mathematics instructional programs should include attention to patterns, functions, symbols, and models so that all students—

- **understand various types of patterns and functional relationships;**
- **use symbolic forms to represent and analyze mathematical situations and structures;**
- **use mathematical models and analyze change in both real and abstract contexts.**

Elaboration for Grades 3–5

Although "algebra" is not a word that has been commonly heard in grade 3–5 classrooms, the mathematical investigations and conversations of students in these grades frequently include elements of algebraic reasoning. These experiences and conversations provide rich contexts for advancing mathematical understanding and are also important antecedents to the more formalized study of algebra in the middle and secondary grades.

In grades 3–5, algebraic ideas emerge and are investigated as students identify or build numerical and geometric patterns, describe patterns verbally and represent them in tables or symbols, look for and apply relationships between varying quantities to make predictions, make and verify generalizations (or rules) that seem to always work in particular situations, use graphs to describe patterns and make predictions, and explore number properties. Students also consider the notion of a variable and begin to use it as a placeholder for representing an unknown or varying quantity.

The Patterns, Functions, and Algebra standard at this grade band includes an important set of mathematical ideas that can engage and stimulate students' thinking in grades 3–5. A variety of examples are used here to clarify the content of this standard at the grade 3–5 level and to show how frequently algebraic ideas arise as students explore and gain facility and confidence in working with number, operations, measurement, data, and geometry.

Focus Areas for Grades 3–5

- **Understand various types of patterns and functional relationships**

  In grades 3–5, all students should—

  - identify, describe, and extend geometric and numeric patterns including growing and shrinking patterns;
  - represent and record patterns using tools such as tables and graphs;
  - identify and describe relationships between two quantities that vary together (e.g., the length of a square and its area);

- investigate and describe situations involving inverse relationships (e.g., the more friends the fewer the cookies for each person, the larger the denominator in a unit fraction the smaller the quantity);
- identify, express, and verify generalizations and use them to make predictions (e.g., doubling a number then doubling again is the same as multiplying by four).

- **Use symbolic forms to represent and analyze mathematical situations and structures**

  In grades 3–5, all students should—

  - identify and use relationships between operations to solve problems (e.g. multiplication as the inverse of division);
  - identify and use algebraic properties of operations to solve problems, e.g., 28 x 7 is equivalent to (7 x 20) + (7 x 8) or (7 x 30) – (7 x 2);
  - develop the concept of variable as a useful tool for representing unknown quantities;
  - use variables (boxes, letters, or other symbols) to solve problems or to describe general rules.

- **Use mathematical models and analyze change in both real and abstract contexts**

  In grades 3–5, all students should—

  - represent and investigate how a change in one variable relates to the change in a second variable (e.g., the height of a plant over time);
  - identify and describe situations with varying rates of change (e.g., a fund-raising effort brought in a small, steady amount of money in the beginning, but more each day as the deadline approached).

Discussion

- **Patterns, functions and relations**

Students in grades 3–5 are encouraged to look for patterns and to express them mathematically (in words or symbols) in order to help them discover, understand, and use properties of algebra to solve a variety of problems. Identifying and working with various patterns helps students develop the ability to classify and organize information systematically and to see that mathematical patterns vary in their structure. Relating patterns in numbers, geometry, and measurement helps them understand connections among mathematical topics. Representing a pattern both geometrically and numerically pushes students in their understanding of the situation and provides multiple perspectives of patterns and relationships. For example, developing and exploring a picture of square numbers (figure 5.3) and expressing the patterns that emerge in mathematical sentences makes properties of these numbers more visible. Students should be encouraged to explain these patterns verbally and to make predictions about what will happen later in the sequence. Students in Grade 4 might use inch graph paper to cut out squares, beginning with a 1-by-1 square, then a 2-by-2 square, and so forth.

With a set of five or six squares they can experiment with their relative size and with ways of representing the patterns they notice in the models.

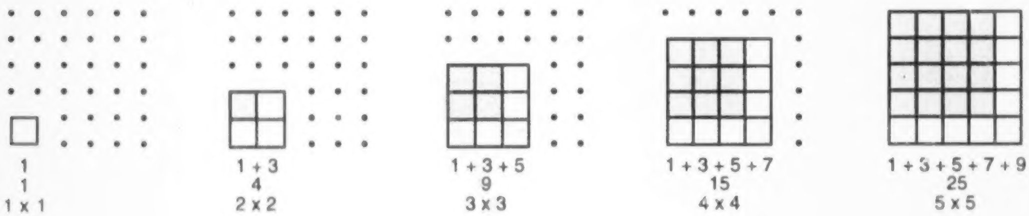


Figure 5.3. Square numbers.

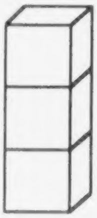
One student noticed that a square always fits into the "corner" of the next larger square leaving an L-shaped empty space. She verbalized this pattern and described the area of the larger square as equal to the area of the smaller plus "two sides" and one more. She represented her solution by writing

Area of a 5x5 square = area of 4 by 4 square +4+4+1

Another student noted the area of the shaded square was increasing by an odd number with each new square. He generated the numbers for each graph below and wrote

Area of a 3 square =1+3+5

Algebraic ideas and reasoning are inherent in the discussions students have as they try to recognize patterns, to describe them, and eventually to represent them. There are many contexts appropriate to grade 3–5 students in which they can investigate and describe relationships between quantities that vary in some predictable way. At first, descriptions can be expressed by using words and later more precisely by functional relationships. As students examine relationships, they make generalizations and predictions. It is appropriate for students in grade three to feel satisfied that they have established a rule for a specific set of examples. By the end of fifth grade, students should be able to justify their generalizations through reasoning about the structure of the relationships. For example, consider the following problem:



What is the surface area of the tower of cubes? As the line of cubes is extended, how does the surface area change?

Figure 5.4. Tower of cubes problem.

A fourth grader might make a table and decide from examining the table that multiplying the number of cubes by 4 and adding 2 yields the number of unit squares "Because it works every time."

Number of cubes N	Surface area (square units) S
1	6
2	10
3	14
4	18

Figure 5.5. Solution to the tower of cubes problem.

A fifth grader should be challenged to justify the rule with reference to the geometric model. For example, "it's four times the number of cubes plus two more because there are always 4 square units around each cube and one extra on each end of the tower" or "it's 5 for the end cubes, then 4 more for all the cubes in between, because the end cubes have 6 but one is covered up, and the cubes in between have 6 but two are covered up." Students might also note the iterative nature of the pattern: "You add four to the previous number." Once a pattern is established students can use it to answer questions like, "What is the surface area of a tower with 50 cubes?" or "How many cubes would there be in a tower with surface area 242 square units?"

In this example, some students may use a table to organize and order their data, whereas others use concrete objects (connecting cubes) to model the growth of an arithmetic sequence. Some students may use words, whereas others use numbers and symbols to express the generalization they find. All these are important and appropriate ways of organizing and expressing ideas about function, and students in grades 3–5 should feel comfortable moving among them.

- **Structure and symbolic notation**

Examples of algebraic properties such as commutativity, associativity, and distributivity of multiplication arise as students work with number and should be the focus of informal discussion and student investigation. Is 5 x 7 the same as 7 x 5? Will such a reversing of factors always result in the same product? What if the factors are larger? What if the factors are decimal numbers? How can you justify that the commutative property holds for the multiplication of any numbers? Students might use an area model to show that 4 x 5 and 5 x 4 have the same physical representation and the same square units of area. The same kind of model can be used later to show that commutativity holds for fractions and decimals as well. Students can also determine whether commutativity is a characteristic of the other operations (addition, subtraction, division).

At times, students use properties quite naturally and intuitively as they compute. For example, a student might multiply 8 and 14 by using the distributive property (8 x 10 plus 8 x 4). The teacher can begin to formalize and extend the student's intuitive understanding of the distributive property. The teacher may want to focus discussion on the process this student is describing. For example, can this technique (breaking one factor apart) be used on other problems, or does it just work with a factor of 14?.Can 14 be broken apart in other ways (e.g., 7 and 7) within this technique? Why might this be advantageous? Can we see why the distributive property is true using an area model?

Discussion about the properties themselves as well as how they serve as tools for solving a range of problems is important if students are to add strength to their intuitive notions. It gives them opportunities to extend and "fill out" their thinking about a strategy and build a foundation for applying it in other situations. Making generalizations about the techniques they employ is a way to extend their mathematical sophistication.

At this grade band the idea and usefulness of a variable (a box, letter, or symbol) should also emerge and be developed. As students explore patterns and note relationships, they should be encouraged to represent their thinking. At first this can be with words ("multiplying the number of cubes by 4 and adding 2 gives the surface area of the tower"), and later students can also record a pattern by using informal or formal variable notation, $4 \times N + 2 = S$. Variable notation is an especially important tool as students begin to work with generalizations.

- **Change and modeling**

Representing and describing change is an important mathematical idea that calls on the tools of algebra. For example, as part of a science project, students have planted seeds and are charting the growth of a plant over time.

Time (Days)	1	2	3	4	5	6	7	8	9	10
Height (cm)	0	0	1	2	4	6	7	8	8	9

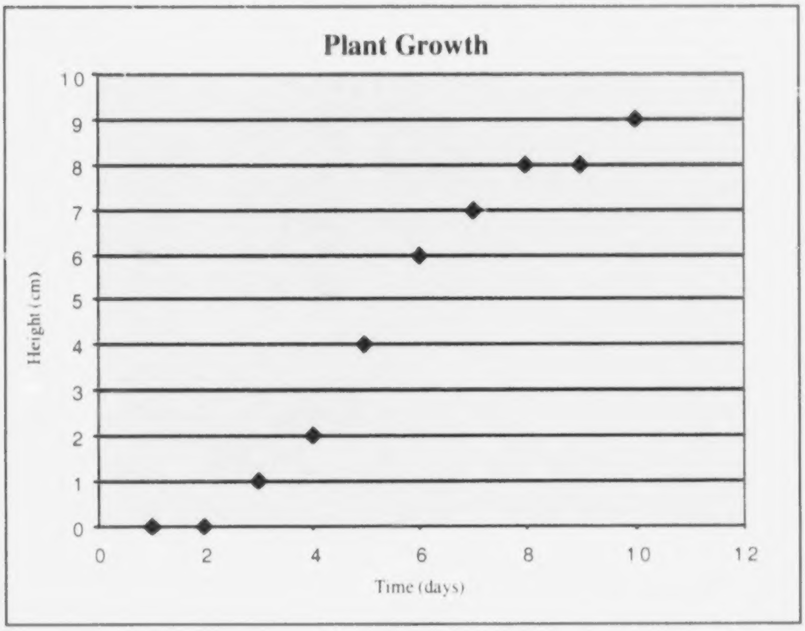


Figure 5.6. Plant growth.

Using the data collected in the table and graph (figure 5.6), students should be asked to describe how the rate of growth varied over time. For example, a student might express the rate of growth in this way: "My plant grew slowly for the first 3 days, then it started to grow faster." They should have many opportunities to collect information and describe the relationships represented in the data—some that vary in a consistent pattern (e.g., surface area vs. number of cubes) and some that vary in non-regular patterns (e.g., the outdoor temperature over a period of the school day).

Standard 3: Geometry and Spatial Sense

Mathematics instructional programs should include attention to geometry and spatial sense so that all students:—

- **analyze characteristics and properties of two- and three-dimensional geometric objects;**
- **select and use different representational systems, including coordinate geometry and graph theory;**
- **recognize the usefulness of transformations and symmetry in analyzing mathematical situations;**
- **use visualization and spatial reasoning to solve problems both within and outside of mathematics.**

Elaboration: Grades 3–5

Geometry includes the study of two- and three-dimensional shapes, properties and relationships of these shapes, and movement and location in space. In the primary grades students construct and examine shapes, describe their attributes, and sort them across various dimensions, such as which three-dimensional shapes roll and which do not, or which four-sided plane figures have square corners and which do not. They begin to develop and use a basic vocabulary related to these shapes but do not develop precise meanings for many of the terms they use.

The growing reasoning skills of students in grades 3–5 allow them to investigate geometrical problems of increasing complexity and to study how geometric terms relate to geometric objects. As they move from grade 3 to grade 5, they develop clarity and precision in describing properties of geometric objects and classifying objects by these properties into categories such as rectangle, triangle, pyramid, or prism. They learn about the relationships between two- and three-dimensional objects by describing the features of three-dimensional objects, representing three-dimensional objects in two dimensions, and constructing or identifying three-dimensional objects by analyzing two-dimensional representations of those objects, such as views, nets, or isometric drawings. They develop knowledge about how geometric shapes are related to each other and begin to articulate geometric arguments about the properties of these shapes. As students investigate geometric properties and relationships, their work is closely connected with other mathematical strands, especially measurement and number.

The study of geometry in grades 3–5, as in pre-K–2, should be active. It requires the use of visual images and concrete materials. Students sort, build, model, trace, measure, and construct. They make and test conjectures and justify their thinking. This exploration requires access to a variety of concrete tools such as graph paper, rulers, compasses, pattern blocks, geoboards, and geometric solids and is greatly enhanced by computer software tools that support exploration.

Focus Areas for Grades 3–5

- **Analyze characteristics and properties of two- and three-dimensional geometric objects**

  In grades 3–5, all students should—

  - identify, compare, and analyze attributes of two- and three-dimensional geometric figures;
  - classify two- and three-dimensional shapes according to their attributes and develop definitions of classes of shapes (e.g., triangle, pyramid);
  - investigate, describe, and reason about the results of subdividing, combining, and transforming shapes using models and representations;
  - investigate and reason about properties of geometric figures (e.g., the number of diagonals of a regular polygon);
  - build and use geometric vocabulary to describe two- and three-dimensional shapes;
  - understand and demonstrate congruence;
  - make conjectures about properties and relationships of shapes and use spatial reasoning and geometric tools to test them.

- **Select and use different representational systems, including coordinate geometry and graph theory**

  In grades 3–5, all students should—

  - describe location and movement using common language and geometric vocabulary (i.e., location of an object in a room or on a grid);
  - set up a coordinate graph (include axes, origin, and scale) and use it to mark and read position using coordinate pairs;
  - make and use coordinate maps to represent actual places (e.g., post office, school, etc.) in an area;
  - represent rectangles and right triangles on a coordinate system and identify vertices with coordinates;
  - find distance between points along horizontal and vertical lines of a coordinate system;
  - given a distance, find pairs of points on the coordinate system separated by that distance.

- **Recognize the usefulness of transformations and symmetry in analyzing mathematical situations**

  In grades 3–5, all students should—

  - predict the results of sliding, flipping and turning two-dimensional figures;

- describe a motion or series of motions that will show that two figures are congruent;
- identify and describe line and rotational symmetry in various two-dimensional shapes;
- explore symmetry in three-dimensional objects.

- **Use visualization and spatial reasoning to solve problems both within and outside of mathematics**

In grades 3–5, all students should—

- identify a three-dimensional shape from a set of two-dimensional views (e.g., match a solid with its front, side and top views);
- visualize and sketch three-dimensional shapes in two dimensions (e.g., make views, nets, or isometric drawings);
- describe and draw geometric shapes from recalled mental images;
- use spatial orientation to navigate to the same point from several different starting positions.

Discussion

- **Shape and structure**

Students in grades 3–5 focus their study of shape on identifying and describing the properties of two- and three-dimensional shapes. In the primary grades, students have begun to build their ideas about what a triangle or a cylinder is, but in these grades, students should develop more precise ways to describe shapes. To consolidate their ideas, students draw and construct shapes, compare and discuss their attributes, classify them, and develop and consider definitions on the basis of a shape's properties (e.g., that a triangle has three straight sides and three angles). For example, many students in these grades will easily name the first two shapes in figure 5.7 as triangles but will need to spend more time discussing why the third one, which does not look so "triangle-like," is nevertheless a triangle. In grades 3–5, students define shapes by describing attributes that are common across all examples of a shape. They also describe attributes do not define the shape, such as its position.

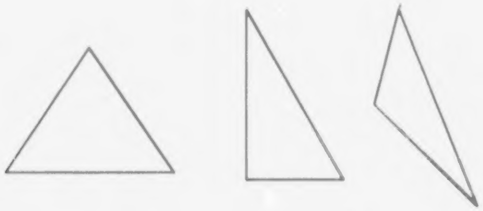


Figure 5.7. Triangles.

Students should also discuss the relationships between classes of shapes. Students need a variety of materials to investigate these relationships—two- and three-dimensional shapes they can examine and compare directly, tools for drawing shapes, and geometric construction software. In such a software environment, students can, for example, construct a parallelogram and investigate how they can manipulate it to make new parallelograms: Which characteristics of the shape change? Which remain the same? Students might examine the two quadrilaterals below by exploring the similarities and differences using dynamic geometry software (Battista 1998). What are the characteristics of each and how are they different?

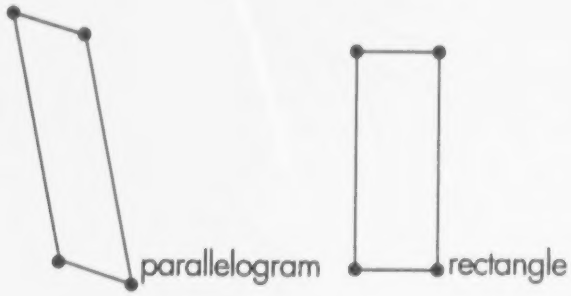


Figure 5.8. Quadrilaterals using dynamic geometry software.

Students in grades 3–5 should be reasoning about geometric relationships within and between shapes. Students should investigate the properties of shapes and make conjectures about these properties. For example, a fourth-grade classroom might consider the following problem:

> Start with any two congruent triangles. Join them along sides of equal length.
>
> What kinds of figures can result?

Students might experiment by drawing various triangles, then tracing or cutting out two copies of each to form figures. This may lead them to the conjecture that parallelograms are made up of two congruent triangles, or perhaps that the opposite sides of parallelograms are congruent. However, if one of the triangles is "flipped over", then a kite will be formed instead of a parallelogram.

In fifth grade, students make frequent use of drawings and concrete shapes to investigate and demonstrate their ideas, but they will also be developing mental models of shapes and their properties. By fifth grade students should be working to develop clear mathematical arguments about why a geometric relationship is true, for example: "You can't possibly make a triangle with two right angles because if you start with one side of the triangle across the bottom, the other two sides go straight up. They're parallel, so they can't possibly ever meet, so you can't get it to be a triangle."

It is critical that students in these grades work with three-dimensional as well as two-dimensional objects. Much of their work with three-dimensional objects will involve describing, comparing, and classifying these shapes. Their work with rectangular prisms is closely connected to their study of volume as they develop the idea of decomposing a three-dimensional shape into cubic units in order to measure its volume.

When describing shapes, students in grades 3–5 are learning to use mathematical terminology by hearing it used repeatedly in context. As students describe shapes, they should hear, become familiar with and use mathematical terms such as parallel, perpendicular, face, edge, vertex, angle, trapezoid, prism, and so forth, to communicate their ideas. Students need to expand their vocabulary so that they can describe geometric ideas with greater precision. For example, to many students, two shapes are the "same" only if they are the same in size, shape, *and* orientation. Thus, a square rotated around its center by 45° is often seen by students as a different shape—a diamond. When students learn the word congruent, their idea of sameness becomes more consistent with what mathematicians mean.

• Coordinate geometry and other systems

In grades 3–5 students build on the idea introduced in pre-K–2 that they can locate points and measure distances by using a coordinate grid. They learn how to use two numbers to name points on this grid and how to navigate on the grid by using landmarks and measurements. Students can map paths between locations, compare lengths of different paths, and examine symmetry, congruence, and similarity of shapes on a coordinate grid. As their ideas about the number system expand to include negative numbers, they can work in all four quadrants (by grade 5). Their work on coordinate graphing should be related to work with other kinds of maps as they recognize that different systems can be used to denote location and to measure distance.

• Transformations and symmetry

Students' study of congruence leads directly to consideration of three important kinds of transformations: reflection, translation, and rotation (flips, slides, and turns). Younger students may have "proved" that two shapes were congruent by physically fitting them one on top of the other, but students in grades 3–5 can develop greater precision in describing the motions needed to show congruence ("turn it 90°" or "flip it vertically"). Although students will not develop a full understanding of similarity until the middle grades when they focus on proportional reasoning, students in these grades can begin to think about the transformations of "blowing up" or "shrinking." When a shape is magnified, what stays the same, what changes?

Students in these grades expand their notion of symmetry. In grades pre-K-2, students focus most often on one line of symmetry in a shape or pattern. In grades 3–5, students explore shapes with more than one line of symmetry. For example, they should explore questions such as these:

- In how many ways can you place a mirror on the square so that what you see looks exactly like the original square? Is this answer true for all squares?
- Can you make a quadrilateral with exactly two lines of symmetry? If so, what kind of quadrilateral is it?

- Can you make a quadrilateral with exactly one line of symmetry? If so, what kind of quadrilateral is it? Can you make a quadrilateral with no lines of symmetry?

Students also describe rotational symmetry more precisely as they learn more about circles and angle measure. Although younger students often create rotationally-symmetric figures with, for example, pattern blocks, they have difficulty describing what they see. They know the pattern repeats in some way, but they are not sure how. In grades 3–5, students can develop descriptions of shapes such as the one in figure 5.9: "it goes out from the middle the same way on every side," "if you turn it 90°, it's exactly the same" or "it would take four turns to get back to where you started, but you can't tell where you started unless you mark it because it looks the same every time."

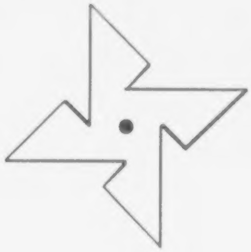


Figure 5.9. Rotational symmetry.

• Visualization and spatial sense

As students in grades 3–5 examine properties and relationships of two- and three-dimensional shapes, they should be challenged to reason about these properties by using spatial relationships (e.g., reasoning about the area of a triangle by visualizing its relationship to a rectangle). In addition to studying actual physical models of these objects, they should also develop and use mental images. Students at this age are ready to mentally manipulate objects and can benefit from experiences that challenge them and can also be verified physically. For example, a student might be asked to imagine and then draw or choose an object with a certain set of attributes (e.g., three sides, one square corner, two sides the same length).

A great deal of the work students do with three-dimensional shapes involves visualization. Through representing three-dimensional shapes in two dimensions and constructing three-dimensional shapes from two-dimensional representations, students learn to describe these shapes with more precision. For example, in order to create a net for a three-dimensional shape, students need to pay attention to the number, shape, and relative position of its faces. Students should become experienced in representing simple three-dimensional shapes in a variety of ways, including freehand drawings, views (i.e., side, front, and top), and nets, and in interpreting two-dimensional representations of three-dimensional objects. For example, can the two-dimensional pattern in figure 5.10 be folded into a cube?

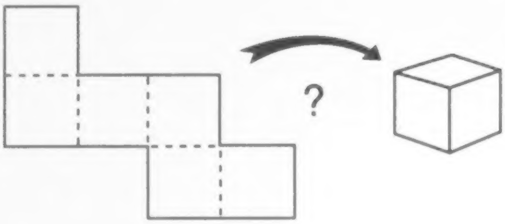


Figure 5.10. Net and cube.

Many aspects of geometry are also closely connected with other mathematical strands, especially measurement. Geometric models can also be used to better understand number concepts. For example, the number line is a model for the set of real numbers, and an area model for multiplication can be used to motivate or explain properties of numbers. Many manipulatives used for modeling number concepts are geometric realizations of arithmetic objects (base ten blocks, fraction pieces).

Standard 4: Measurement

Mathematics instructional programs should include attention to measurement so that all students—

- **understand attributes, units, and systems of measurement;**
- **apply a variety of techniques, tools, and formulas for determining measurements.**

Elaboration: Grades 3–5

Measurement is a process that students and adults use every day. Using concepts and tools of measurement, data can be collected and a variety of techniques can be used to describe and quantify the world. In school mathematics programs, measurement helps connect ideas within areas of mathematics and between mathematics and other disciplines. It can serve as a context to help students introduce and develop important mathematical concepts such as fractions, decimals, and proportional reasoning.

Measurement is both an object of study (including concepts, skills, and strategies) and a process to be applied. In learning about measurement and learning how to measure, students should be actively involved, drawing on familiar and accessible contexts. Students in grades 3–5 should measure objects and space in their classroom, use maps to determine distance around their community, determine an appropriate unit and use it to measure the area of their classroom, estimate the time it takes to do various tasks, and measure and represent change in thesize of attributes, such as their height or that of plants they are growing.

Prior to grade 3 students develop a general understanding of the process of measurement—identifying an attribute to be measured, choosing an appropriate unit, and iterating that unit in order to assign a number to a particular attribute of an object or event. They should also have had many experiences measuring different attributes, particularly length, weight, and time. In grades 3–5 students expand their knowledge of and facility with measurement by examining additional attributes (e.g., area, volume, and angle size), paying closer attention to degree of accuracy, and using a wider variety of measurement tools for obtaining measures. They also begin to develop and use formulas for measuring certain attributes (e.g., perimeter or area).

Focus Areas for Grades 3–5

- **Understand attributes, units, and systems of measurement**

  In grades 3–5, all students should:

  - develop awareness of measurements as approximations and understand how the tools used to measure affect the level of precision;
  - understand the need for uniform units of measurement and develop facility in using the common units of the English and metric systems of measurement;
  - identify attributes such as length, area, and volume and know the type of unit and tool needed to measure each attribute;
  - carry out simple unit conversions within a system of measurement (e.g., centimeters to meters, hours to minutes);
  - explore the effect on size when objects change under simple transformations (e.g., do different shapes made from linking eight cubes have the same surface area? the same volume?).

- **Apply a variety of techniques, tools, and formulas for determining measurements**

  In grades 3–5, all students should:

  - select and apply appropriate standard units (metric and English) and tools (e.g., rulers, grid paper, graduated cylinders) to estimate and measure length, area, volume, weight, time, temperature, and angles;
  - determine the perimeter, area, and volume of shapes and solids by counting segments, square units, or cubic units;
  - develop, understand, and use formulas to find the perimeter and area of rectangles;
  - estimate measurements of physical objects by comparing the objects to benchmarks (e.g. estimate the size of an angle by relating it to a right angle);
  - determine the surface area of cubes and other rectangular solids by considering each face;
  - use map scales to measure distance between locations and make simple scale drawings using grid paper.

Discussion

- **Attributes, units, and systems of measurement**

Students in grades 3–5 continue to measure attributes of a variety of physical objects, extending to more complex attributes including area, volume, mass (weight), and angle. They explore attributes of area and volume by using nonstandard units (e.g., beans) to cover or fill up a region or space. They learn that length measurements in particular contexts are given specific names, such as perimeter, width, height, circumference, and distance. They begin to establish some benchmarks by which to estimate or judge the size of other objects. For example, they learn that a "square corner" is called a right angle and establish this as a benchmark against which to compare other angles.

Students in grades 3–5 should recognize the need to select units appropriate to the attribute being measured. Different kinds of units are needed for measuring area than for measuring length. At first they use any objects that fill space to estimate area and come to recognize the need for a standard unit such as a square centimeter to cover space. Likewise, the need for a different kind of unit (cubic unit) to measure volume grows out of experiences that begin with filling containers (e.g., with beans or rice or water) and considering the unsuitability of such units (they don't completely fill the space).

Students in grades 3–5 encounter the notion that measurements in the real world are approximate, because of the instruments we use and our ability to interpret the results. For example, in measuring objects using a centimeter ruler, we are limited in the precision of the measurement by our eyesight and by the gradations on the ruler. Students encounter this as they compare measurements for various objects with one another. For example, the activity in figure 5.11 provides a context in which the teacher raises and the class considers measurement as an estimation process.

Measure and Compare: Work in pairs and use your rulers to measure the items on the chart posted at the front of the room. Record your measures for each object on the chart.

OBJECT TO MEASURE	Jo & Rustin	Whit & Beth	Ben & Anna
Height of teacher's desk	70 cm	68 cm	67 cm
Diameter of clock face	92 cm	96 cm	91 cm
Length of classroom	8 m	7.5 m	8.2 m

Figure 5.11. Measurements.

Each pair of students found slightly different measurements even though the groups were measuring the same object. Since the item being measured (e.g., the diameter of the clock) didn't change, what leads to slightly different measurements? Students' responses will vary according to their age, experience, and mathematical maturity. In grade 5, teachers should hear explanations that include the degree of precision used by the students (how precisely did they read the scale on the instrument used?), the scale provided by the instrument used (is it marked in centimeters or millimeters?), and the students' perceived need for accuracy (how much precision is needed in this particular context?). The discussion might lead to considering the importance of measuring precisely in certain contexts (e.g., carpenters often measure "twice" and use special instruments in order to minimize waste of materials) and being quite satisfied with an estimate in other instances (e.g., the Scout troop hiked about 2.5 miles).

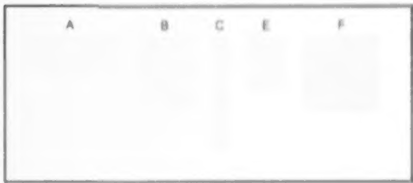
- **Techniques, tools and formulas for measurement**

Students in grades 3–5 expand the number of tools and the range of techniques they use to measure. They move toward using conventional tools, such as rulers and tape measures, for measuring length and need instruction to learn to use these tools properly (e.g., reading the markings on a ruler or identifying the amount of weights on a pan balance). When students have difficulty using available tools to measure a particular attribute, they can learn to invent techniques that will work in a given situation. For example, a rule is not helpful in measuring the circumference of a clock face, but students might use string to outline the clock face and then measure the string.

Students should be challenged to adapt measurement techniques as needed in order to measure complex figures or objects. For example, they might measure the area of an irregular polygon by covering it with transparent grid paper and counting units, or by breaking it apart into regular shapes that they can measure. They might measure the volume of a small irregular container by immersing it in water in a graduated cylinder and seeing how high the water rises.

Students in grades 3–5 should be encouraged to use their growing understanding of units of measure and attributes to make measurement estimates, oftentimes using chunking or benchmark strategies. For example, to estimate the length of the classroom, students might use a tile as a unit of measure (a "chunk" of the object to be measured) and decide how many tiles make up the length. Then the task becomes one of estimating the length of the tile and multiplying that value by the number of tiles. Another strategy for estimating measurements is to compare the item to be measured against some benchmark. For example, a student might estimate the teacher's height by noting that the teacher is about half again as tall as the student. They know their own height and can use it to estimate the height of the teacher. Strategies for estimating measurements are varied and often depend on the particular situation. They provide a nice opportunity for students to share thinking and techniques. Students also need experience in judging what degree of accuracy is required in a given situation, and whether an underestimate or overestimate is desirable. If a person is buying carpet, error should be in the direction of an overestimate. However, an estimate of the amount of time to warm up a food item should probably be an underestimate.

Formulas for measuring perimeter, area, and volume should emerge as students have opportunities to measure, record information, and look at patterns. For example, the following table (figure 5.12) was generated by a group of third graders who used a transparent grid to determine the area of a set of rectangles. They found the grid an easy tool to use to determine the areas of several rectangles. As they generated the table they realized that counting all the squares wasn't necessary once the length and width of the rectangle were determined. They tested their conjecture (that Area = $L \times W$) using the grid, and it appeared to work for each rectangle in the set. Later, their teacher challenged them to think about whether their formula would always work—for big rectangles as well as small ones.



Rectangle	Length	Width	Area
A	5	2	10 sq,
B	3	4	12 sq,
C	1	6	6 sq,
D	2	3	6 sq,
E	4	4	16 sq,

Figure 5.12. Areas of rectangles.

Teachers should capitalize on opportunities to use measurement to link mathematics to science and other subjects. For example, students can explore science concepts by investigating the relationship between the drop height and bounce height of different balls, as in the following problem:

> Ball Bounce Experiment. Do all balls have the same amount of "bounce"? Think about a tennis ball or a racquetball. Do they bounce the same? If not, how can you describe the amount of bounce a ball has? One way is to compare the bounce of a ball to the height from which it was dropped. Try this experiment: Drop several different kinds of balls from different heights. Make a chart of the drop and bounce height. Can you make any predictions?

BALL TYPE	BOUNCE HEIGHT	DROP HEIGHT	RATIO
Tennis ball	53 cm	100 cm	About 5/10
Racquetball	73 cm	100 cm	About 7/10
Softball	37 cm	100 cm	About 4/10
Golfball	62 cm	100 cm	About 6/10

Figure 5.13. Bouncing balls.

As students do this experiment and collect the data, as in figure 5.13, they will find that really bouncy balls bounce almost back to where they were dropped. Not-so-bouncy balls don't bounce back that far. The ratio of bounce height to drop height (B/D) is a derived measure that we can call the "bounciness" of the ball. A tennis ball usually has a bounciness ratio around 1/2, while a racquetball's bounciness ratio is closer to 7/10.

Measurement activities are more engaging and thought provoking if they are carried out for a purpose. Students encounter the need for measurement through activities such as the "ball bounce" above and as they explore questions related to their class or school environment. For example:

- How long does it take members of the class to get ready for school each morning?
- How much water is consumed at the school water fountain each day?
- How much catsup is used in the school cafeteria each day?
- What is the area of the school playground? How much space does that allow each student if all fourth graders are on the playground at the same time?
- What is the distance from my house to school?

In summary, measurement is an important strand in grades 3–5, because of its practicality, its utility in linking mathematics to the real world, and its connections to other areas of mathematics.

Standard 5: Data Analysis, Statistics and Probability

Mathematics instructional programs should include attention to data analysis, statistics, and probability so that all students—

- **pose questions and collect, organize, and represent data to answer those questions;**
- **interpret data using methods of exploratory data analysis;**
- **develop and evaluate inferences, predictions, and arguments that are based on data;**
- **understand and apply basic notions of chance and probability.**

Elaboration: Grades 3–5

Students in grades 3–5 are becoming more aware of the world beyond themselves, their families, and their classrooms and are beginning to think about larger issues and questions. Students at these levels are ready to expand their knowledge about how to investigate, describe, and analyze information about their world, using data analysis as a tool. Through collecting, representing, and interpreting data, students use statistical approaches to organize and make sense of information.

Investigations involving data should happen frequently during grades 3–5. These can range from brief class surveys to investigations that take many days. Frequent, short data investigations (How many brothers and sisters do people in our class have? What's the farthest you have ever been from home?) can be used to acquaint students with the many aspects of collecting, representing, summarizing, comparing, and interpreting data (i.e., the class takes fifteen minutes to gather data, post it on a line plot, and analyze the results). More extended projects can engage students in a cycle of investigation. They formulate questions, collect, represent, and analyze initial data, then consider whether their data are giving them the information they need. They might then represent their data in a different way and consider other statistical descriptions, or they might revise their question or experiment and collect and analyze new data. Students learn about revision and presentation to an audience: they can develop their own rough drafts of graphs and tables as they analyze their data and then create more polished representations when they are ready to present their data.

In these more extended projects, students should sometimes address questions and experience investigations that have the potential to result in conclusions or recommendations that could influence decisions. For example, one class investigating playground injuries at their school gathered evidence that led to the realization that the bars on one piece of playground equipment were too large for the hands of most students below third grade. This finding resulted in a new policy for playground safety.

Students in grades 3–5 begin to actively consider the likelihood of events. At these levels, informal explorations involving probability are appropriate. Such explorations typically involve the discussion of the likelihood of events, both in familiar contexts (how likely is it that it will snow in July in our town?) and in experiments (if I put all the class names in a box, how likely is it that we'll pick one that starts with L?).

Focus Areas for Grades 3–5

- **Pose questions and collect, organize, and represent data to answer those questions**

  In grades 3–5, all students should—

  - formulate questions they want to investigate;
  - design data investigations to address a question;
  - collect data using observations, measurement, surveys, or experiments;
  - organize data using tables and graphs (e.g., bar graph, line plot, stem-and-leaf plot, circle graph, and line graph);
  - use graphs to analyze data and to present information to an audience;
  - compare data representations to determine which aspects of the data they highlight or obscure.

- **Interpret data using methods of exploratory data analysis**

  In grades 3–5, all students should—

  - describe the shape and important features of a set of numerical data, including its range, where the data are concentrated or sparse, and whether there are outliers;
  - describe the center of sets of numerical data, first informally, then using the median;
  - classify and describe categorical data (e.g., ways we travel to school) in different ways; analyze and compare the information highlighted by different classifications;
  - compare related data sets, with emphasis on the range, center, and how the data are distributed;
  - propose and justify conclusions based on data;
  - formulate questions or hypotheses based on initial data collection, and design further studies to explore them.

- **Develop and evaluate inferences, predictions, and arguments that are based on data**

  In grades 3–5, all students should—

  - describe how data collection methods can impact the nature of the data set;
  - discuss the concept of representativeness of a sample within the context of a particular example (e.g., is the class representative of other fifth-grade classes in our town? In the U.S.? In Canada? Why or why not?);
  - compare the data from one sample to other samples and consider why there is variability;
  - in simple experiments, infer the structure of the population through drawing repeated samples (with replacement).

- **Understand and apply basic notions of chance and probability**

  In grades 3–5, all students should—

  - discuss events as likely or unlikely and give descriptions of the degree of likelihood in informal terms (e.g., unlikely, very unlikely, certain, impossible);
  - estimate, describe, and test probabilities of outcomes by associating the degree of certainty with a value ranging from 0 to 1 (e.g., in simple experiments involving spinners with different fractions shaded).

Discussion

- **Collect, organize, and represent data**

Activities involving data at these grade levels should involve students in responding to questions about a variety of situations that interest them. These may include determining a favorite music group or carnival ride, surveying the class or school about the amount of television watched each day, or completing an experiment to determine how far students can throw a softball. Once a question is formulated, data can be collected. Data then can be presented in charts, tables, and graphs. Each form of representation is appropriate for certain kinds of questions and data, and each highlights particular aspects of the data set. Students should become familiar with a variety of representations by creating them, watching their teacher create them, and considering representations found in the media. Students need to consider aspects of their representations that help people understand them: the headings for tables, the names for axes on a graph, the items labeled on a chart, or the scales used in a graph. As they construct graphs of data they have collected, they work to understand what the values along the horizontal axis mean (e.g., the categories of the ways they get to school or the number of people in their families) and what the values along the vertical axis mean (the frequency with which those values occur in the data set). Through experience with a variety of graphs, students encounter and discuss issues such as why the scale on the horizontal axis needs to include values that are not actually in the data set and how to represent zero (0) on a graph.

Students should become familiar with quick ways to represent their data, such as tables or line plots, so that they can get an initial, overall look at the data, for which a bar graph will be constructed and displayed during a class presentation. Students will also explore when a line graph may be more appropriate to use than a bar graph; and when stem-and-leaf plots and circle charts may be informative or more efficient when presenting their data.

- **Describe, analyze and interpret data**

In the pre-K–2 years, students are often most interested in data about themselves or in which value is "the most" on a graph. In grades 3–5, students begin to be able to regard a set of data as a whole, which can be described as a set and compared to other data sets. For example, it begins to make sense to students at this level to discuss a "typical" value for a set of data and to be able to describe the shape of a distribution. It is critical that students in these grade levels engage in description and analysis that moves beyond what they undertook at the pre-K–2 grades. They need to become used to looking at data in more than one way and in considering several of its characteristics.

Students in grades 3–5 encounter different kinds of data—data that can be categorized and data that can be ordered numerically. As students examine a set of ordered numerical data, they learn to pay attention to characteristics of the data set: where data are concentrated or clumped, where there are no data, or which data points appear to have unusual values. For example, a fourth-grade class plotted the heights of "Fast Plants" grown in their classroom (figure 5.14) (Wisconsin Almuni Research Foundation 1989, p. 11). Students describing these data should notice that the lowest height is 15 cm and the tallest plant is 40 cm; most of the data are concentrated from 20 to 23 cm; and the plant that grew to a height of 40 cm is very unusual, far removed from the rest of the data. Students gradually learn statistical terms such as range and outlier that help them describe the set of data.

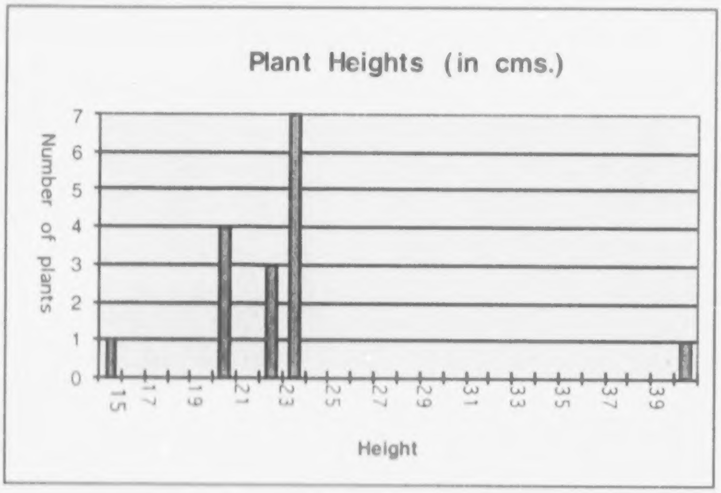


Figure 5.14. "Fast Plants" bar graph.

With categorical data students should try different classifications to get different views of the data. For example, a simple survey about types of pets becomes more complex as students examine different categorizations; one group might classify the pets as mammals, fish, amphibians, and so forth; another group might classify them according to whether they were strays, purchased at a pet store, progeny of friends' pets, and so forth. Students might use fractions and percentages to describe their findings.

Comparing data sets leads to the need to discuss how data sets are similar or different. Figure 5.15 shows stem-and-leaf plots of the heights of Fast Plants grown in two classrooms (Wisconsin Alumni Research Foundation 1989, p. 11). Students noticed that although the ranges are about the same, the data from the fourth grade are concentrated between 20 and 23 cm and the third-grade data are much more spread out, with many more higher values.

Heights of Fast Plants (in centimeters)

Fourth Grade	Third Grade
0|	0| 9
1| 5	1| 47
2| 00002223333333	2| 23556777899
3|	3| 0123359
4| 0	4| 0

Figure 5.15. "Fast Plants" stem-and-leaf plot.

During grades 3–5, students gradually develop the idea of a "typical" or average value. Building on students' understanding of "the most" and "the middle," they can learn about three kinds of average - mode, median, and, informally, the mean. Students need to learn more than simply how to identify the mode or median in a data set. They need to build an understanding of what that number tells them about the data, and they need to see those values in the context of other characteristics of the data. In figure 5.15, the median of the fourth-grade data is 22.5 but the median of the third-grade data is 27.5, indicating that half the third-grade data must be equal to or greater than 27.5. But it is also important to look at the distributions of the data, which show that almost 80 percent of the third grade's plants are taller than all but one of the fourth grade's plants. Towards the end of these grades, once students are experienced using the mode and median as part of their data descriptions, they can begin to conceptually explore the role of the mean as a balance point for the data set, using small data sets.

- **Inference and prediction**

In these grades, students should engage in comparing several, related data sets (e.g., the heights of students in each grade), and begin to consider more than one variable in a data set (e.g., how height and age are related). As they encounter comparisons and associations, they think both about the characteristics of the data and about why those characteristics may exist. Just as in other areas of mathematics, there is an emphasis here on the use of evidence as validation. As students discuss their data, they begin to distinguish between what the data actually show and what might account for their results. For example, a fourth grade class found that the median Friday bedtime for boys was an hour and a half later than for girls. (Russell et al. 1995) After describing their data, they began to develop hypotheses about the reasons for this difference: Did girls also get up earlier than boys, so that they get the same amount of sleep? Were girls more compliant while boys spent more time arguing, thereby delaying their bedtimes?

Toward the end of grades 3–5, students begin to understand that data sets are samples of larger populations. They can look at several samples drawn from the same population (e.g., family size in several classrooms in their school) or compare their own sample to known statistics about the population (e.g., the median family size for their class with the statistic reported for their town). They can think about the issues that affect the representativeness of a sample and begin to notice how samples from the same population can vary.

- **Probability**

Students in grades 3–5 encounter beginning ideas about how probability can be used to describe the likelihood of events. These ideas are advanced by actively exploring familiar events in everyday contexts, particularly explorations involving chance. For instance, what is the likelihood of seeing a commercial when you turn on the television? Students can collect data about the number of minutes of commercials in an hour to estimate this probability.

Students can also experiment with probability in situations with only a few outcomes, such as game spinners with certain portions shaded. At this level, probability is linked closely to understanding important ideas involving rational numbers. Students can consider how the likelihood of an experimental event can range from impossible to certain and can denote this probability range as 0 to 1.

In summary, students in grades 3–5 develop some of the critical skills of data analysis. At these levels data analysis becomes a vehicle in which to investigate interesting and important questions. Students expand their repertoire of data representations (tables, graphs, and other representations); they learn about statistical measures that help them describe and summarize data; they compare related data sets; and they develop language to discuss their informal notions of probability.

Standard 6: Problem Solving

Mathematics instructional programs should focus on solving problems as part of understanding mathematics so that all students—

- **build new mathematical knowledge through work with problems;**
- **develop a disposition to formulate, represent, abstract, and generalize in situations within and outside mathematics;**
- **apply a wide variety of strategies to solve problems and adapt the strategies to new situations;**
- **monitor and reflect on their mathematical thinking in solving problems.**

Problem solving is the process of identifying and using current understanding to formulate or adapt strategies to address a new situation. Problem solving should be the central focus of the mathematics curriculum. As such, it is the primary goal of all mathematics instruction and an integral part of all mathematical activity. Problem solving is not a distinct topic but a process that should permeate the entire program and provide the context in which concepts and skills can be learned (NCTM 1989, p. 23.)

What does problem solving look like in grades 3–5?

When faced with new mathematical situations that do not conform exactly to previously learned techniques, students must develop and use strategies—they must solve problems. A question such as, "How can you share nine brownies among eight people?" is a true problem for many students in Grades 3–5, who do not have learned computational strategies for this problem. However, students have access to skills and concepts that can aid them in solving the problem.

According to Hiebert et al. (1997), students develop mathematical understanding, "as they invent and examine methods for solving mathematical problems" (p. 17). A good mathematics problem generally has three features. First, it is problematic; there is something to make sense of and the route to a solution is not readily apparent. Second, it is challenging and interesting from a mathematical perspective. Third, it connects with knowledge the student already has so that current knowledge and skills can be adapted and applied to complete the task. Good problem tasks should encourage reflection and communication and can emerge from the students' environment or from purely mathematical contexts. Students build new mathematical understandings through work with problems. The substance of daily mathematics lessons can effectively be approached by working through genuine problems.

Good problem-solving tasks generally serve multiple purposes, such as challenging students to develop and apply strategies, introducing them to new concepts, and providing a context for practicing skills. For example, consider the following problem posed by a teacher who wanted to involve students in a discussion of probability concepts:

> If you roll two dice and multiply the numbers that land face up, is it more likely that the resulting product will be odd or even?

Initially, students thought that the even and odd products would be equally likely. As they generated data by rolling the dice several times, they were surprised by evidence showing the likelihood of even products. After additional trials, the teacher asked them to make conjectures about why even products were more likely, and several explanations emerged. Some students made or referred to multiplication tables and began to count the number of odd and even products. Others observed, "Any time you multiply a number by 2, it always gives you an even answer. It's true when you multiply by 4, too." Other students began to explain the general case. "An odd times an odd is always odd. An even times an even is always even. When you have an odd times an even you get an even answer." In order to extend the activity the teacher asked students to consider how much more likely it was to get an even product than an odd one. Students were surprised to discover that they were three times more likely to get an even product than an odd one. This activity illustrates a situation where a simply-stated question generated a considerable amount of inquiry. It provided an opportunity for students to use data collection techniques, caused them to examine organized representations of data in order to make sense of outcomes, and allowed them to practice multiplication facts while exploring new mathematical content.

Students must also understand that problems can be solved in many different ways and that various strategies provide different insights. One teacher asked students to develop several different ways to determine which fraction was larger, 3/7 or 5/9. Students described the following strategies:

> "Draw a good picture of each fraction then compare pictures."
>
> "Compare each fraction to 1/2. Since 3/7 is less than 1/2 and 5/9 is greater than 1/2, 5/9 must be larger."
>
> "Write each fraction in equivalent form using a common denominator then compare them."
>
> "Subtract each fraction from one whole, that's 4/7 and 4/9, then compare the size of these."
>
> "Build a pattern (1/5, 2/6, 3/7, 4/8, 5/9, 6/10, . . . 96/100) and see what's happening (the fractions are getting larger)"

In thinking of several different ways to compare the two fractions, students made connections with various mathematical concepts and had opportunities to hear about other strategies that might not have occurred to them. The problem also provided many rich opportunities to analyze the relationships among the strategies and evaluate them in terms of efficiency, accuracy, and generalizability (e.g., drawings can help provide information but may not be reliable in making decisions about fractions very close in size).

Good problems help encourage thinking about generalized rules. Using this pattern of dots in figure 6.15, students were asked to find several ways of determining how many dots were on the boundary of the square and then to represent their thinking and their solutions in a way that would be understandable to others.

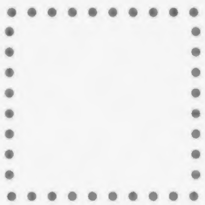
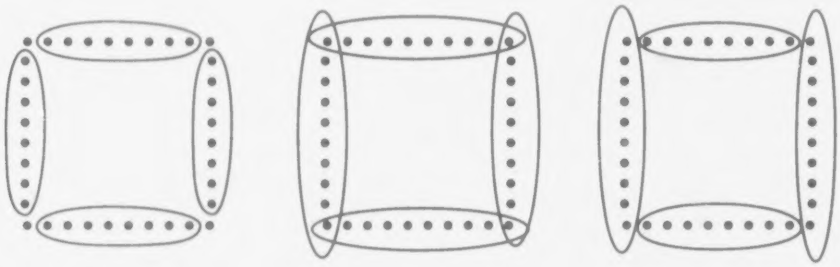


Figure 5.16. The dot squares problem.

When students presented their solutions to the class, as in figure 5.17, the teacher asked them to relate the drawings to the numbers in their equations. When all strategies had been presented, the teacher asked students to examine the various ways of solving the problem and to notice how they were alike and how they were different. The teacher then extended the problem by asking students to determine the total number of dots if there were 98 on each side of the square, including the corners, and to decide which strategy would be best to use in answering the question. Thus, students were asked to evaluate the effectiveness of the strategies for a class of problems.



$4 \times (10 - 2) + 4 = 36$ $10 \times 4 - 4 = 36$ $10 + 8 + 10 + 8 = 36$

Figure 5.17. Student work on dot squares problem.

In addition to developing and using a variety of strategies, students also need to learn how to ask questions that extend problems. In this way, students are encouraged to follow up on their genuine curiosity about mathematical ideas. For example, the teacher might ask students to create a problem similar to the dot square problem or to extend it in some way. For example, "If there were a total of 76 dots, how many would be on each side of the square?" or "could there be a square formed by a total of 75 dots?" By extending problems and asking different questions, students become problem posers as well as problem solvers.

What is the teacher's role in helping students in grades 3–5 become better problem solvers?

The teacher's role in helping students become problem solvers is to establish the classroom environment, select rich and appropriate tasks or problems, orchestrate their use, and assess students' understanding. Critical to the development of confident, self-assured problem solvers are the social norms established in the classroom by the teacher. Students must be taught to respect and value one another's ideas and ways of thinking. These attitudes are essential if students are expected to make sense of mathematics, take intellectual risks by raising questions and formulating conjectures, and support their ideas with mathematical arguments (NCTM 1991, p. 57).

In this environment the teacher supplies students with a series of problems or tasks worthy of intellectual inquiry and consistent with the curriculum. They must also provide students with time to explore problems thoroughly; it is far better to explore a few problems deeply than many superficially. In addition, the teacher must be sensitive to opportunities for mathematical connections. During a science unit on conservation, one teacher asked students whether they thought the hallway water fountain was a resource-efficient device for distributing water. Students decided to estimate how much water was wasted at the drinking fountain. They took several measurements showing how much water the fountain released in a minute and averaged the results. They followed a similar procedure for averaging students' use of the fountain. Then they had to figure out ways of determining how much water was actually swallowed and how much went down the drain. They decided to time several students and measure water as they drank from a glass. Speculating that drinking from the fountain would be slower, they estimated water consumption and formulated a response concerning the efficiency of a water fountain. In this case, an investigation was prompted by a simple question that the students found interesting.

Since good problems challenge students to think, they will often struggle to arrive at solutions. This is not bad. It is the teacher's responsibility to know when a particular student needs assistance or is able to continue working, despite frustration. Giving help too soon can deprive students of the opportunity to make mathematical discoveries. Students need to know that a challenging problem will take some time and that perseverance is an important aspect of problem solving.

When students share their solutions with classmates, it is critical that teachers help them probe various aspects of their strategies. Teachers cannot be satisfied with explanations that are simply procedural descriptions or summaries. Rather, they must expect students to support explanations with mathematical reasons (Kazemi 1998). In the dialogue below, a teacher questions two fourth graders as they describe how they divided nine brownies equally among eight people:

Sarah: The first four we cut them in half. [Jasmine divides squares in half on the overhead.]

Ms. Carter: Now as you explain, could you explain why you did it in half?

Sarah: Because when you put it in half it becomes four . . . four . . . eight halves.

Ms. Carter: 8 halves. What does that mean if there are eight halves?

Sarah: Then each person gets a half.

Ms. Carter: Okay, . . . each person gets a half. [Jasmine labels halves 1-8 for each of the eight people.]

Sarah: Then there were five boxes [brownies] left. We put them in eighths.

Ms. Carter: Okay, so they divided them into eighths. Could you tell us why you chose eighths?

Sarah: It's easiest. Because then everyone will get . . . each person will get a half and [addresses Jasmine, "How many eighths?" Jasmine replies quietly back to her, "5/8."]

Ms. Carter: I didn't know why you did it in eighths. That's the reason. I just wanted to know why you chose eighths.

Jasmine: We chose eighths because then if we did eighths, each person would get each eighth, I mean 1/8 out of each brownie.

Ms. Carter: Okay, 1/8 out of each brownie. Can you just, you don't have to number, but just show us what you mean by that? I heard the words, but . . .

Jasmine: [Shades in 1/8 of each of the five brownies not divided in half.]

Jasmine: Person one would get this . . . [points to 1/8]

Ms. Carter: Oh, out of each brownie.

Sarah: Out of each brownie, one person will get 1/8.

Ms. Carter: One eighth. Okay. So how much then did they get if they got their fair share?

Jasmine & Sarah: They got a half and 5/8.

Ms. Carter: Do you want to write that down at the top, so I can see what you did?

Jasmine: [Writes 1/2 + 1/8 + 1/8 + 1/8 + 1/8 + 1/8 at the top of the overhead.]

In this dialogue, the teacher pressed students to give reasons for their decisions and actions: What does that mean if there are eight halves? Could you tell us why you chose eighths? Can you… show us what you mean by that? She was not satisfied with a simple summary of the steps, but instead expected students to give verbal justifications all along the way and to connect those justifications with both numbers and graphics. Eventually, these students are likely to imitate their teacher's questioning techniques in their own small-group discussions. In this way students learn how to monitor and reflect on their mathematical thinking in problem situations.

Listening to discussions, the teacher is able to assess students' understanding. In the brownie dialogue, the teacher knows the students have found a way to divide the brownies equally. If she questions the students further by asking if each person received more or less than one brownie (and how they can justify their response), she may gain additional information about their conceptual knowledge. For any assessment of problem solving, teachers must look beyond the answer to the reasoning behind the solution. This evidence can be found in classroom dialogue and in written explanations and drawings. Reflecting on this assessment data, the teacher has some direction for future instruction.

By creating a supportive classroom environment, carefully selecting tasks for students, sensitively implementing their use in the classroom, and continually assessing students' understanding of mathematical concepts, the teacher assists students in becoming problem posers and solvers.

Standard 7: Reasoning and Proof

Mathematics instructional programs should focus on learning to reason and construct proofs as part of understanding mathematics so that all students—

- **recognize reasoning and proof as essential and powerful parts of mathematics;**
- **make and investigate mathematical conjectures;**
- **develop and evaluate mathematical arguments and proofs;**
- **select and use various types of reasoning and methods of proof as appropriate.**

Elaboration: Grades 3–5

Reasoning is fundamental to knowing and doing mathematics. Mathematical reasoning in grades 3–5 involves the kind of thinking, conjecturing, and validating that helps students see that mathematics makes sense. In these grades, students are learning fundamental ideas about the nature of mathematics as a logical system. Therefore, these years are an pivotal time in the development of students' understanding of what it means to do mathematics. A focus in these grades is for students to create mathematical ideas themselves and to articulate, examine, and evaluate these ideas.

During grades 3–5, students are involved in a key transition in their mathematical reasoning. Many students begin this grade band believing that something is true because it has occurred before, because they have seen several examples of it, or because their experience to date seems to confirm it. During these grades, they learn to assess claims on the basis of an examination of evidence. Students learn that several examples are not sufficient to establish the truth of a conjecture. They learn that they need to consider a range of possible examples and to reason about the general properties and relationships they find across examples.

A classroom climate that encourages students to put forth their own ideas for examination is very important to the development of mathematical reasoning. Both teachers' and students' statements should be open to questions, reactions, and elaborations from others in the classroom. Students need to know that being able to explain and justify their thinking is important, and should begin to learn how to detect fallacies in others' thinking. This mind-set is established when they have opportunities to apply their reasoning skills and when justifying their thinking is an expected component of mathematics discussions. Students need a great deal of time, many experiences, and a good model and guide to develop their ability to construct valid arguments and evaluate the arguments of others.

What does reasoning look like at grades 3–5?

Students in grades 3–5 should be actively thinking about mathematics as a discipline in which the work involves making conjectures, developing mathematical arguments about these conjectures, and attempting to verify or disprove the arguments. In Ms. Taylor's third grade class, students are having a discussion of the problem 4 x 8. During the discussion, students talk about how they mentally computed. One student, Matt, explained, "I thought of

2 x 8, that's 16, then you just double it." The teacher pauses and calls on other students to restate Matt's idea, then asks, "Do you think Matt's way of multiplying by 4 always works?" When it's clear that the response is quite mixed, she asks the students to try a few problems like this themselves before gathering again to discuss Matt's conjecture.

This example shows a teacher taking advantage of an opportunity to engage her students in mathematical reasoning. By asking the question, "Do you think that always works?," she moves the discussion from the specific problem to a consideration of a general characteristic of multiplication problems—that the factors of a multiplication problem can themselves be factored then multiplied (said another way, the associative property holds for multiplication). In this specific case, Matt has noticed that multiplying by four is the same as multiplying by two then by two again. In this classroom, mathematical reasoning is a central part of all mathematical activity. Students are expected to think beyond finding the solution to an individual problem to an understanding of mathematical structures and relationships.

After students had worked on several problems and had discussed with a partner why "doubling then doubling again" was a strategy for multiplying by four, the teacher reconvened the class for further discussion. Student responses showed a wide range of thinking in the group:

Carol: Because if you have 2 times 8 and 4 times 8, you're doubling the answer. It works every time.

Leonard: If you have 2 times 8, it would be 16, so if you want to get to 4 times 8, you're adding two more of the 8's, so it's doubling.

Malia: It has to be doubled because you're doing the same thing over again. It's like you did 2 times 8 is16 and then you did 2 times 8 is 16 again, so it has to be 32.

Steven: What you're doing is counting by eights, so you're counting ahead, you're skipping some of the eights. You're doing another two of them, so it's like doubling them up.

Matt: I tried to see if it would work with triples, so I did 2 times 8 and 6 times 8, and it worked. You times it by three and the answer is tripled.

At this point, the teacher sent students off to work on their set of multiplication problems, knowing she would return to this discussion again. During their work on the problems, several students used ideas from the class discussion. One girl solved 74 x 6 by starting with 74 x 2, then doubled the problem and the answer, then added the two products together to get the final product (see figure 5.18).

74 x 2 = 148

74 x 4 = 296

148 + 296 = 444

Figure 5.18. Solution to 74 x 6.

What do students learn about mathematical reasoning in grades 3–5?

Students in grades 3–5 should frequently make conjectures about mathematical objects and relationships, investigate those conjectures, and make mathematical arguments that are based on their work. Students need to know that posing conjectures and trying to validate them is an expected part of their mathematical activity. In order to do this, students must develop a range of reasoning skills. They learn to generate examples to test their conjectures. They learn to search for possible counterexamples. They learn that conjectures posed on the basis of preliminary evidence may not hold.

An important mathematics instructional goal at these grades is coming to understand what constitutes a convincing mathematical argument. For example, in a grade 4 classroom, students were solving the following problem:

> Fold a rectangle in half, divide one half into two congruent right triangles, divide the other half into two congruent rectangles (Figure 5.19). Then compare one of the right triangles and one of the small rectangles; do they have the same area or does one have a larger area than the other?

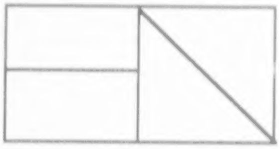


Figure 5.19. Folding rectangles.

Many students at first tried to solve this problem by "eyeballing" the figure:

> The triangle is bigger because it goes way up;
>
> No, the rectangle looks a teeny bit bigger.
>
> I think they're the same because the triangle's taller, but the rectangle's longer.

As the students worked on this problem, some students were convinced that the figures were equal (or not) depending on whether they could cut up the triangle and fit it on the rectangle in a disorganized way, whereas others thought about how to organize their cutting and pasting, for example, by cutting the triangle into two pieces to make it into a congruent rectangle. Still others developed ways to reason about the relationships in the figure without cutting and pasting: "We folded the paper in half, so the part with the rectangles is the same as the part with the triangles, and so they have to be the same." All students were quite convinced by their own arguments.

At this grade level, many students were just beginning to develop an idea about what a convincing argument is. The first solution—cutting and pasting in a disorganized way—focuses only on these particular shapes: is there a way to make one fit on the other? The second solution takes into consideration some of the characteristics of the triangle and rectangle: the triangle can be rearranged in an organized way to make a rectangle with the same area. While the students who used this solution may not yet have been generalizing to any right triangle, their method opens up that possibility. The third solution is the beginning of reasoning about relationships and structure, although the student has not yet developed an articulate argument to explain why one of the triangles has the same area as one of the rectangles.

As in the example above, there is a range of what grades 3–5 students understand verification to be. Some students are satisfied if they can show the example (or several examples) to be true. For the case of the "double-double" strategy to multiply by four, some of the third graders argued that their conjecture was true because "it works every time." From grades 3 to 5, students should move towards verification that depends on reasoning about relationships and properties. Students need to be questioned and challenged with questions such as, "What if I gave you twenty more problems like this to do—would they all work? How do you know?" Through comparing solutions and questioning each other's reasoning, students begin to learn to describe relationships that hold across many instances and to craft arguments about why those relationships can be generalized and to what cases they do and do not apply.

Sometimes even students who seem to have developed a clear argument about a mathematical relationship need to be questioned and challenged when they are ready to encounter new aspects of the relationship. For example, a class of third graders had spent a great deal of time working with arrays in their study of multiplication. As a group, they were very sure that multiplication was commutative and they could demonstrate this property using arrays. However, in fourth grade they began encountering larger numbers, and when the teacher noticed that some students were using commutativity as they solved problems, she brought up this notion with the class and challenged them to develop an argument that would prove it would hold for any numbers. It turned out that when she pressed, "But would it work for any numbers? How about 43,279 times 6,892?" they lost their confidence that commutativity is a property of all multiplication problems. They could no longer use a physical array to model such large numbers, and they needed further work to develop mental images and mathematical arguments based on what they had learned from the physical model. These students will also need to revisit this idea when they study computation with fractions and decimals.

Sometimes students' conjectures about mathematical properties and relationships will turn out to be wrong. It is essential for students to learn that part of mathematical reasoning is examining and trying to understand why something that looks like it might be true is not and to begin to use counterexamples in this context. Perhaps even more important, they should come to understand that when one is engaged in mathematical reasoning, coming up with ideas that turn out not to be true is part of the endeavor. These "wrong" ideas often serve to surface important mathematical discussions and discoveries.

As students develop new mathematical skills and knowledge, or encounter new kinds of mathematical problems, they need to continually revise, expand, and update generalizations they have made. When students apply their reasoning in an area of mathematical content, they begin to develop a web of generalizations that provides a foundation for mathematical understanding and insight (Russell, 1999). As Matt, the student in the doubling example, started thinking beyond doubling to tripling, he was beginning to reason about a larger class of problems and might eventually come to a more general statement of his conjecture.

What is the teacher's role in developing mathematical reasoning?

In order for mathematical experiences such as those in the examples in this section to happen frequently, the teacher must establish the expectation that the class as a mathematical community is continually developing, testing, and applying conjectures about mathematical relationships. For example, in a fourth-grade classroom, students were ordering fractions. To begin this activity, the teacher had asked students to identify fractions that are more than 1/2 and less than 1. After students talked in pairs, the teacher asked how they were choosing their fractions:

Patrize: We were talking about how you could get it, and if you make the top number, the numerator, higher than a half of the denominator, but you don't make it the same as the denominator like 5/5 'cause then it will be a whole.

Teacher: It sounds like you have a conjecture. Can someone else explain it?

Justin: Like if you have 3/4, half of 4 is 2, so you want the number higher than 2 but not 4.

Teacher: One more person explaining it?

Rita: 4/8 would be the same as 1/2 cause 4 is half of 8, so anything above the half would work, but it can't be the same number or it would be a whole.

By questioning students in this way routinely, the teacher is establishing the expectation that students are to listen carefully and try to understand other students' ideas. In classrooms where reasoning takes place, communication—both stating ideas clearly and listening carefully—must also be a priority. Besides listening and restating ideas, as above, students need to learn during these grades to develop sound mathematical arguments. They must view their job as crafting arguments that are adequate and convincing.

The teacher will also have to make decisions about which conjectures are mathematically significant for the students in that classroom to pursue. In order to do this, the teacher must take into account the skills, needs, and understandings of the students and the teacher's mathematical agenda for the class. Finally, the teacher should work to continually remind students of conjectures they have come up with and mathematical arguments they have crafted to apply to further work. Reasoning is not something to be done on particular days in mathematics class but should be woven throughout all the mathematics in which students are engaged.

Standard 8: Communication

Mathematics instructional programs should use communication to foster understanding of mathematics so that all students—

- **organize and consolidate their mathematical thinking to communicate with others;**
- **express mathematical ideas coherently and clearly to peers, teachers, and others;**
- **extend their mathematical knowledge by considering the thinking and strategies of others;**
- **use the language of mathematics as a precise means of mathematical expression.**

Elaboration: Grades 3–5

The ability to read, write, listen, think, and communicate about problems will develop and deepen students' understanding of mathematics (NCTM 1989, p. 78). In grades 3–5, students use communication as a tool for understanding and generating solution strategies. Writing is more coherent than in earlier grades and mathematical terminology is often used along with everyday language to explain concepts. In classroom discussions, students become the audience for one another's comments--that is, they speak to one another, aiming to convince or question their peers. Above all, the discourse is focused on making sense of mathematical ideas and on using mathematical ideas sensibly in setting up and solving problems (NCTM 1991, p. 45).

In grades 3–5 students expand their abilities to learn from and work with others. Their communication consists not only of conversations between student and teacher or one student with another student, but of students listening to a number of peers and joining group discussions in order to clarify, question, and extend conjectures. Depending on the purpose for writing, students also become more detailed and coherent in their descriptions of problem solving strategies and reasoning.

What does communication look like at grades 3–5?

In grades 3–5, classrooms communication (sharing thinking, asking questions, explaining, and justifying) should be a common occurrence. The classroom environment should encourage students to express and write about their mathematical conjectures, questions, and solutions. For example, after preparatory work in decimals, Ms. Nolte engaged her students in the following problem in order to help them use their understanding of decimals to think about methods for adding decimals (Schifter, Bastable, and Russell in press, pp. 82-91):

> Pretend you are a jeweler. Sometimes people come in to get rings resized. When you cut down a ring to make it smaller, you keep the small portion of gold in exchange for the work you have done. Recently you have collected these amounts:
>
> 1.14g .089g .3g
>
> Now you have a repair job to do for which you need some gold. You are wondering if you have enough. Work together with your group to figure out how much gold you have collected. Be prepared to show the class your solution.

In this activity, the teacher put the students in a problem solving situation. Although they had worked with representing decimals, they had not discussed adding them. As was customary in the class, students were expected to talk with their peers to solve the problem and to share their results and thinking with the class. Students used communication as a natural and essential part of the problem-solving process. As the groups worked, the teacher circulated among the students:

Nicole: We could line the numbers up on the right like you do with other numbers.

Ned: Maybe we should line up the decimals, but I don't know why we would do that.

Teacher: I think you're suggesting that you might line this problem up differently from the way you line up whole-number addition. Is that right?

Ned: (Nods.)

Teacher: Why do you line whole numbers up the way you do? What's the reason for it?

Ned: I don't know. It's just the way you do it. That's how we learned to do it.

Mark: I think it would help if we drew a picture, like of the [base ten] blocks. Maybe we could figure it out then.

The teacher then moved to another group where the students have represented their problem like this:

$$\begin{array}{r} 114 \\ 089 \\ +\quad 3 \\ \hline 206 \end{array}$$

Teacher: What happened to the decimal numbers?

Joe: We just decided to drop the decimals and add the numbers like usual. That way we could line them up on the right and add. We left the zero in there, but you can just leave it out since it doesn't mean anything.

Teacher: Do you all agree?

Students: Yes.

Teacher: Are you saying, then, that if you start out with one and fourteen hundredths grams of gold and some other little bits that it adds up to two hundred and six grams of gold?

Listening carefully to the group discussions, Ms. Nolte rephrased Ned's suggestion in order to make sure she had accurately captured Ned's thinking, to help him focus on the important mathematical concepts, and to guide him in considering how this problem is related to those more familiar to him. His response gave her important assessment information about

Ned's understanding of the addition algorithm. Although he was able to follow the procedures of addition, he lacked an understanding of the concepts behind the procedures. Placed in the problem situation of figuring out how to add decimals, he needed to deepen his understanding of whole numbers as well. In talking with Joe's group, the teacher asked a question that led the students to think about the reasonableness of their response by considering it in relation to its real-world context. The realization that their response didn't make sense caused the students to revisit the problem. In this particular instance, the teacher chose to let the children work through their confusion. The teacher's decisions about what to say or not say, what to ask or not ask, were based on her observations of how students have worked on this problem, what they have done in previous work, her interpretations of their discussions, as well as on her desire to nudge students to do further critical mathematical reasoning.

After the groups finished their work, the class as a whole had a discussion. Rob reported that the children in his group represented the problem in this way:

$$\begin{array}{r} .3g \\ 1.14g \\ +\ .089g \\ \hline 1.529g \end{array}$$

Ned immediately asked how they had decided to line up the numbers that way and Rob responded that the group thought they needed to line up the tenths with the tenths and the hundredths with the hundredths to make it come out right. Joe, whose group got two and six hundredths for their answer by dropping the decimals, speculated that it was possible to drop the zero since in .089, since "it doesn't stand for anything." Teresa immediately jumped in with, "You can't just drop that zero. It has to be there or you get 89 hundredths instead of 89 thousandths, and they're not the same at all." Mark continued to push for a model, but he was stumped. "If I had the flats be one whole, then the rods are tenths and the units are hundredths, but I don't know how to draw the thousandths except as dots. Then I can't really tell what's going on." Another student, Ben, then suggested that the *thousand* block be one whole, so the flats could be tenths, the rods hundredths, and the units thousandths.

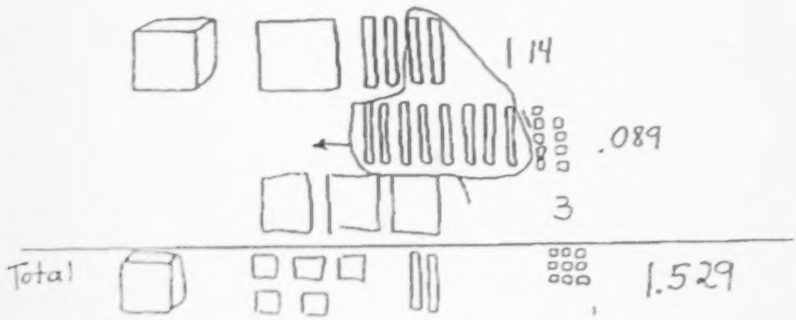


Figure 5.20. Ben's work.

Mark paid close attention during Ben's explanation, nodding that he understood. Teresa was also interested in the explanation, especially realizing that the zero was very significant in deciding which blocks to use. After this presentation, students wrote in their journals. Many mentioned that the demonstration had made it clear that tenths had to be added to tenths and hundredths to hundredths for the right answer. Some made up new problems and made drawings of the base ten block model. Students who had eliminated the decimal point wrote "Mess Up Tips" explaining that their approach was the wrong way.

Because these students were in an environment where student talk about thinking was a regular occurrence, children were comfortable in articulating their ideas, even if they were different from those of their peers. Besides focusing on their own thinking, students also attempted to understand the thinking of others and in some cases to relate it to their own. Ned, who earlier had been unable to articulate why whole numbers are lined up as they are in the traditional addition algorithm, questioned Rob about why his group had lined up the numbers. Ned was taking responsibility for his learning by asking questions about a concept that wasn't quite clear to him. Ben thought about Mark's dilemma and came up with a solution that became clear to Mark.

The use of the base-ten block model provided a further opportunity for understanding and conversation. Having a concrete referent to point toward helps students have understandings that are clearer and more easily shared. (Hiebert et al. 1997.) The talk that preceded, accompanied, and followed Ben's presentation gave meaning to the base-ten model. Mark had been "stuck" by viewing the model in one way until Ben showed him another way to look at it.

Throughout the lesson, the dynamic interactions in this classroom were critical aspects of the children's ability to make sense of what they were doing. Because there was time to talk, write, model, and draw pictures as well as occasions for work in small groups, large groups, and as individuals, students who worked best in different ways all had opportunities to learn.

What is the teacher's role in stimulating and improving communication?

Teachers must create classroom environments in which intellectual risks and sense making are expected. Teachers must also routinely provide students with rich problems centered on the important mathematical ideas in the curriculum so that students are working with situations worthy of their conversation and thought. In daily lessons teachers must make on-the-spot decisions about which points of the mathematical conversation to pick up and which to let go, and when to let students struggle with an issue and when to give direct instruction. Ms. Nolte, for example, chose to let one group of students struggle with the fact that their answer was unreasonable within the context of the problem. Teachers must refine their listening, questioning, and paraphrasing techniques, both to direct the flow of mathematical learning and to provide models for student dialogue. Well-posed questions can simultaneously elicit, extend, and challenge students' thinking. Periodically, teachers may need to explicitly discuss students' effective and ineffective communication strategies. Teachers can model questioning and paraphrasing, for example, then point out and explain those techniques to their students. Teachers can also highlight examples of good communication among students. ("I noticed that Karen and Malia disagreed on an answer. They not only explained their reasoning to each other very carefully, but they listened to one another. Both understood each other's reasoning. It was hard, but eventually, they realized that one way made more sense than the other.") Teachers need to help students learn not to criticize, but to ask questions when they disagree or do not understand a classmate's reasoning. When making a concept or strategy clear to a peer, the student-explainer is forced to reexamine and thus deepen mathematical understandings. In settings where communication strategies are taught, modeled, and expected, students will eventually begin to adopt listening, paraphrasing, and questioning techniques in their own mathematical conversations.

Teachers also need to provide students with some assistance in writing about mathematical concepts. Teachers should expect students' writing to be correct, complete, coherent, and clear. Especially in the beginning, teachers will need to send writing back to the students to be revised. Students will also need opportunities to check the clarity of their work with peers. Initially, when students have difficulty knowing what to write about in mathematics classes, the teacher might ask students to use words, drawings, and symbols in order to explain a particular concept to an absent classmate. To help students write about their reasoning processes, the teacher could pose a class problem solving activity and later ask, "What have you done so far to solve this problem, what decisions did you make, and why did you make those decisions?" As students respond, the teacher could explain, "This is exactly what I'd like you to tell me in your writing." Sharing student work is another way to help students understand what is expected, including what complete and incomplete responses look like. For example, students were asked to write a complete answer to the following question (Kouba, Zawojewski, and Strutchens 1997):

> Jose ate 1/2 of a pizza. Ella ate 1/2 of another pizza.
>
> Jose said that he ate more pizza than Ella, but Ella said they both ate the same amount.
>
> Use words and pictures to show that Jose could be right.

Student responses reflected different levels of understanding, as in figure 5.21. The first student assumed that each pizza was the same size. Although the student used words and drawings in the response, the answer was correct only if the unit was the same, an assumption that can't be made from the statement of the problem. The second student indicates by his drawing that the size of 1/2 depends on the size of the unit. However, he neglects to describe his thinking. The third solution, including written words and drawings, is correct and complete in that it communicates why Jose could be correct. Discussion of various student responses throughout the year, especially as mathematical concepts and problems become more complex, is an effective way to help students continue to improve their ability to communicate.

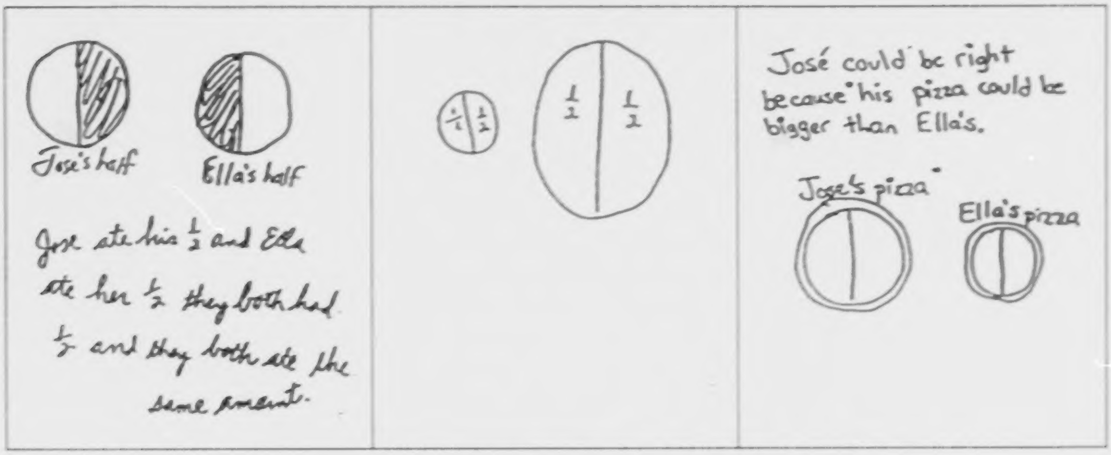

Jose's half
Ella's half
Jose ate his 1/2 and Ella ate her 1/2 they both had 1/2 and they both ate the same amount.
1/2
1/2
1/2
1/2
José could be right because his pizza could be bigger than Ella's.
Jose's pizza
Ella's pizza

Figure 5.21. Student solutions for the pizza problem.

Standard 9: Connections

Mathematics instructional programs should emphasize connections to foster understanding of mathematics so that all students—

- **recognize and use connections among different mathematical ideas;**
- **understand how mathematical ideas build on one another to produce a coherent whole;**
- **recognize, use, and learn about mathematics in contexts outside of mathematics.**

Elaboration: Grades 3–5

Mathematics consists of a web of related ideas. Although mathematics is sometimes thought of as a set of isolated topics, procedures, and processes, the power of mathematics lies in the connections of one idea to another. These connections are many and varied within the grades 3–5 curriculum. For example, the operations of multiplication and division are related in that one is the inverse of the other. Multiplication can be thought of as repeated addition. It can be modeled through an array and is thus linked to the concept of area of a rectangle. The multiplication table suggests a host of patterns and can lead to investigations of prime and composite numbers. Knowing the value of 5 x 9 enables students to develop methods for computing 5 x 90 or 50 x 900, or even 500 x 18 (e.g., double 500 x 9). Over time and with appropriate experiences and guidance, students in grades 3–5 should continually look for, discover, discuss, and utilize links between mathematical ideas and processes. With such ongoing investigations and conversations, students can begin to see the underlying structure and beauty of mathematics.

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Why are mathematical connections important at grades 3–5?

Students in grades 3–5 are meeting a considerable amount of new mathematical content, and their ability to understand and manage these new ideas will rest in part on how well the ideas are connected. Real-world contexts provide many rich and varied opportunities for students to link what they are learning to the world around them and vice versa. Students' experiences at home, school, and in their community provide sites for worthwhile mathematical tasks. These sites and opportunities should be pursued to take advantage of built-in student interest. For example, a group of fifth graders might want to investigate questions regarding the school lunch menu—How many students in the school generally eat school lunch on pizza day? Is this number consistent across the year? Does it vary by the day of the week pizza is served or by the season of the year? How do the school lunch organizers estimate the number of pizzas they will need? Encouraging students to ask questions and facilitating discussion helps students see the value of mathematics and also motivates the study of new mathematical ideas.

Mathematics investigations without a real-world context are also appropriate and important. For example, in the section of this chapter on representations, a student investigation of the following problem is discussed: "In the multiplication table, are there more even or odd products? Why?" This question is a mathematical one, without a real world connection, but it is motivating to students in that it presents an intellectual challenge—one that can be investigated by grades 3–5 students using a variety of approaches. The value of a mathematical task is not dependent on whether it has a real-world context, but rather by whether it is intellectually engaging and solvable using tools the learner has or can draw on.

In addition to connections between and among mathematical ideas, mathematics is connected to and utilized within other subjects. Building on these connections provides opportunities to enrich the learning in both content areas. For example, in an economics unit in social studies, a fourth-grade class was discussing the population and size (land mass) of each state. They investigated the following question: "Which state has the most land per person living in the state? Which state has the least amount of land per occupant?" Students used almanacs and maps and constructed charts to summarize their findings. The project provided many opportunities for them to read about and discuss issues of population density and land management and to use mathematical skills to analyze and organize the information.

In grades 3–5, connections to language arts (written and oral communication and reading) play an increasingly important role in learning and assessment. When students are reading about a meaningful situation, they can use context and other cues to help with understanding. Additionally, many problem-based curricula require students to read and assimilate printed information. Some teachers acknowledge that, while this creates some tensions, it also contributes to improving students' reading ability. At grades 3–5, where attention to the development of language and reading skills is prominent, it makes sense to ensure and capitalize on connections to mathematics. For example, the linkage of language to mathematics through the use of appropriate children's literature is a good way to launch mathematical discussions. After reading *How Much is a Million* (Schwartz 1985), students may discuss their own experiences with "big numbers" or plan a collection campaign.

Science and mathematics are closely connected in that it is natural for content within one area to emerge as investigations in the other area are underway. In grades 3–5 students are developing important processes needed for scientific inquiry and for mathematical problem solving—inferring, measuring, communicating, classifying, and predicting. The kinds of investigations that enable students to build these processes often include significant mathematics as well as science content. It is important that teachers talk about and promote discussion about both the mathematics and science ideas that emerge from the investigations, whether they occur within a science class or within a mathematics class.

What is the teacher's role in helping students make mathematical connections?

Teachers should select tasks that help students explore and develop increasingly sophisticated mathematical ideas. They should recognize and use situations that link mathematics and other subject areas. They should ask questions to encourage and challenge students to build arguments and develop strategies based on the mathematics they know. For example, asking students, "Does that answer seem reasonable to you?" should encourage students to focus on the answer and the situation and to draw on other knowledge to justify their thinking or to question their answer.

Teachers should encourage students to be curious about their world and to make use of mathematics to understand it. Scientific contexts can be especially productive for exploring and using mathematics. Teachers should build on everyday experiences to encourage the study of mathematical ideas through systematic, quantitative investigation of phenomena that students can relate to or are interested in. These may include applications as varied as studying the relationship between arm span and height, investigating the strength of a particular brand of paper towel, or studying how the sizes of different cereal boxes vary and how size affects the amount of cereal contained in the box. Mathematical contexts can be equally compelling for grades 3–5 students as sites for exploring mathematical connections. Students might look for patterns in "super even" numbers (doubles of even numbers), create graphs, and find geometric representations.

Connecting mathematical ideas helps students see mathematics as a unified body of knowledge rather than as a set of complex and disjoint concepts and procedures. If students view mathematical ideas as connected to each other, they are more likely to work toward understanding it, rather than simply memorizing facts without understanding. It is the task of teachers and curriculum developers to create tasks for students that have solid mathematics content and that significantly increase student's knowledge and abilities and continue to build the web of linked mathematical ideas.

Standard 10: Representation

Mathematics instructional programs should emphasize mathematical representations to foster understanding of mathematics so that all students—

- **create and use representations to organize, record, and communicate mathematical ideas;**
- **develop a repertoire of mathematical representations that can be used purposefully, flexibly, and appropriately;**
- **use representations to model and interpret physical, social, and mathematical phenomena.**

Elaboration: Grades 3–5

Mathematical ideas or situations can be represented in a variety of ways—through a concrete physical model, an equation, a graph, or a mental picture. The very act of representing a problem or idea requires a student to articulate and clarify the idea and give it structure so they (and sometimes others) can evaluate and extend it. Mathematicians rely on representational tools as a language for communicating and thinking. Students in grades 3–5 need to extend their ability to create and use representations as they encounter and explore mathematical concepts and tools and use them to solve problems.

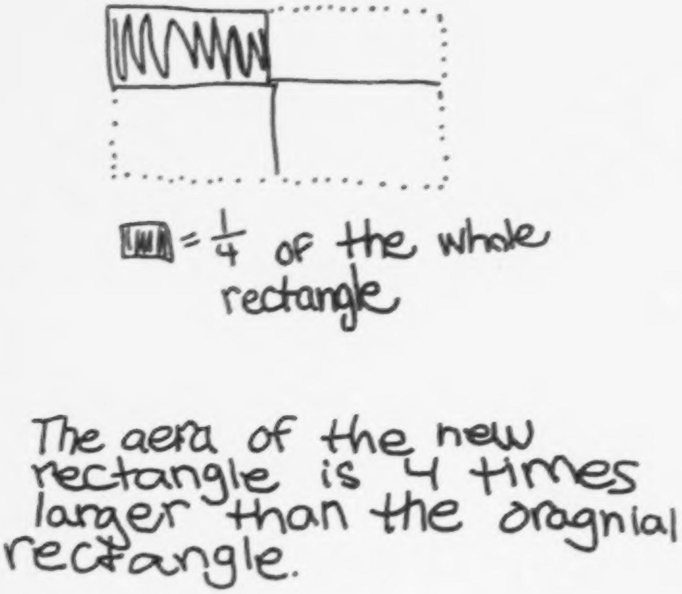
What is mathematical representation and why is it important?

A representation is an image. This image may be a physical model: a table, diagram or graph, or perhaps even a written response to a problem. A mathematical representation may also take the form of a mental image of a problem—one that can be quickly referred to but does not need an external form. Such representations help to portray, clarify, or extend a mathematical idea by focusing on certain essential features. Students represent ideas when they create a table of data about a favorite sports team, when they describe (in words or in a picture) the important features of an object such as a cylinder, or when they translate a problem into a number sentence, equation, or chart. Good representations fulfill a dual role as an instrument of thought and as an instrument of communication (Esty and Teppo 1994; Kaput 1989).

Consider the following problem that a class of fifth-grade students is asked to think about:

> What happens to the area of a rectangle if the lengths of its sides are doubled?

Students who represent the problem in some way are more likely to see important relationships than those who consider the problem without a representation. Whitney's initial response to the problem was that the new rectangle would be twice the size of the first rectangle. Whitney had her own mental image of the problem, but it was not established through any real experience with a physical representation. She might have stopped there, but a fellow student questioned the answer and this prompted Whitney to "defend" her response—in reality to think more deeply about the question. She decided she needed a picture (Figure 5.22) to prompt her own thinking.



▨ = $\frac{1}{4}$ of the whole rectangle

The aera of the new rectangle is 4 times larger than the oragnial rectangle.

Figure 5.22. Whitney's representation.

Whitney's drawing helped her better understand the complexity of the problem and prompted her to consider it more carefully. In particular, her picture showed her that the new rectangle is not only bigger, but that it is four times bigger than the original rectangle. It also provided a way to show her answer and to defend it to others.

Betsy approached the same problem by listing values for various rectangles and looking for a numerical pattern. After generating a few examples, such as 5 x 5 = 25 and 10 x 10 = 100, she concluded that the areas were quadrupled. Her examples were limited to special cases (squares). She didn't directly consider the more general question. Students need to learn that representations that are limited to special cases can be misleading. The teacher plays a particularly important role here to prompt more inquiry and to point out inaccurate or incomplete representations. The examples Betsy generated helped her see that the new area is four times the original area. However, it didn't clearly show her why. Whitney and Betsy represented the mathematical problem in different ways. Both ways provided a representation that allowed them to observe how doubling the length and width affected the area of the rectangle. Different representations clearly highlight different features of a situation, and students will benefit from experiences depicting a mathematical situation in multiple ways.

Representing numbers by using different mathematical equations provides a means for thinking about numbers in different ways and exploring number properties. For example, a third-grade class of students worked together to generate ways to think about and represent the number 36 (see figure 5.23).

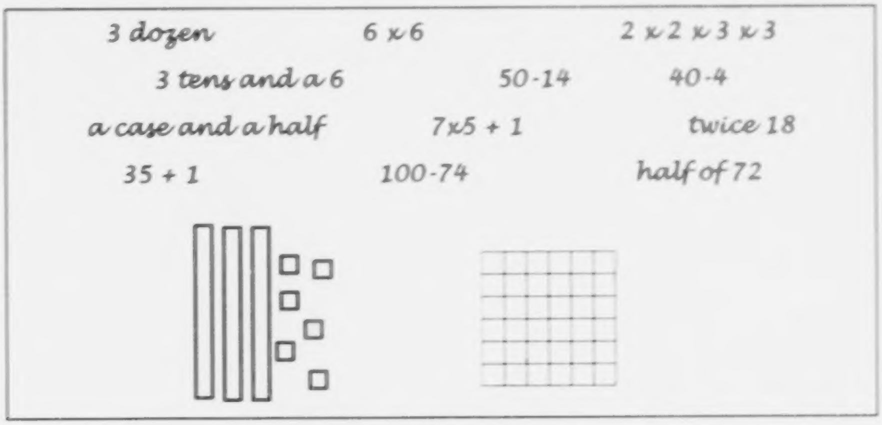


Figure 5.23. Ways to represent 36.

Each representation of 36 highlights different properties of the number and can be useful for different purposes. Seeing 36 in these varied ways expands a student's way of viewing and using numbers. It also reinforces the notion that different mathematical expressions conjure up new or different images. For example, 3 tens and 6 is a way to think about 36 that may be helpful in adding 36 to another number. On the other hand, thinking of 36 square meters as the size of a 6 by 6 square room may be helpful as a benchmark for thinking about the size of another room or space. Students who view numbers in multiple ways have more tools for thinking about and using them.

What is the teacher's role in developing and using representations?

In grades 3–5 students acquire and use increasingly sophisticated and varied tools for representing mathematical ideas, including techniques for organizing charts and graphs, and using physical models such as arrays and geometric solids. They begin to explore the value of multiple representations of the same idea. In addition, they use tools such as calculators and computer software to represent and organize information and ideas. They need to be involved in experiences which help them do the following:

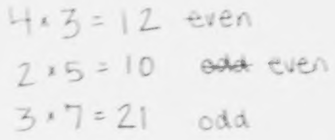
- record mathematical thinking in an organized way;
- create accurate and complete representations of mathematical ideas;
- use conventional notational schemes to explore and communicate new ideas of mathematics;
- investigate why some representations are more useful than others in certain situations.

Learning to record or represent thinking in an organized way, both in solving a problem, and in sharing a solution, is an acquired skill for many students. Teachers can and should emphasize the importance of representing mathematical ideas in a variety of ways.

Modeling this process as they work through a problem with the class is an easy and effective way to stimulate some students to take your lead. Talking through why some representations are more beneficial than others also gives prominence to the process and helps students critique various representations. For example, consider the following question, which a fourth-grade class explored:

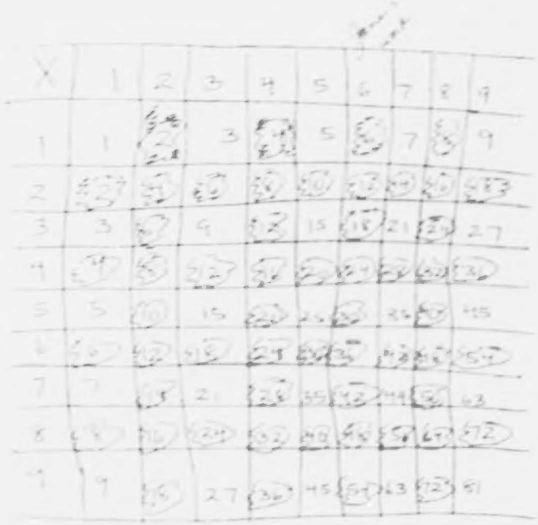
> Are there more even or odd products in the multiplication table? Explain why.

Some students may initially generate examples to formulate an answer (figure 5.24). Other students might make a more systematic list such as Jane has done with a multiplication table (figure 5.25). Organizing the work helps highlight patterns that otherwise are obscured. The teacher may want to represent the idea in another way (figure 5.26) in order to encourage generalization.



4×3 = 12 even
2×5 = 10 ~~odd~~ even
3×7 = 21 odd

Figure 5.24. Scott's work.



X	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	10	12	14	16	18
3	3	6	9	12	15	18	21	24	27
4	4	8	12	16	[illegible]	24	[illegible]	[illegible]	36
5	5	10	15	20	25	[illegible]	35	[illegible]	45
6	6	12	18	24	[illegible]	36	42	48	54
7	7	14	21	28	35	42	49	56	63
8	8	16	24	32	40	48	56	64	72
9	9	18	27	36	45	54	63	72	81

Figure 5.25. Jane's work.

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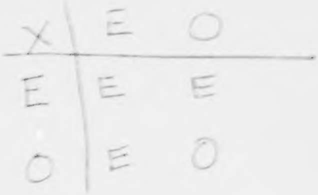
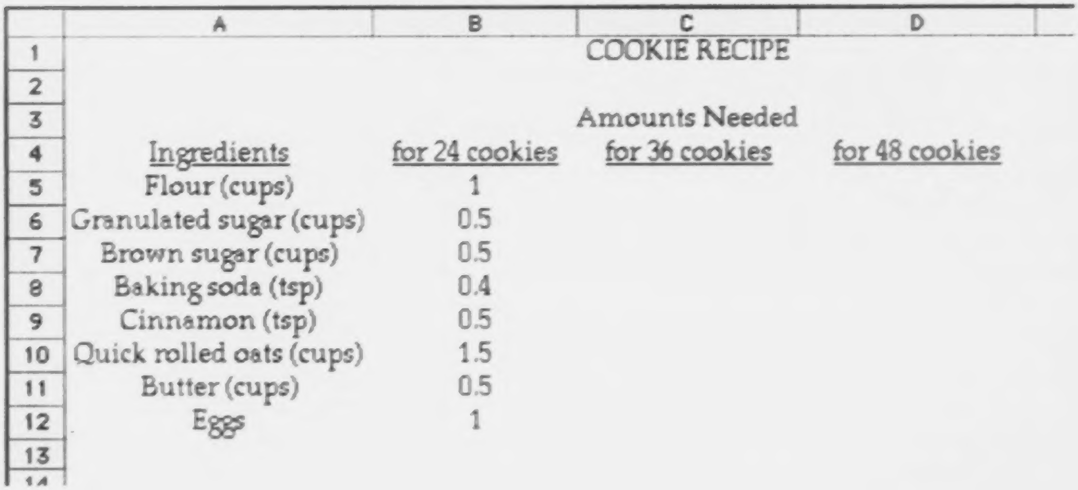


Figure 5.26. Teacher's work.

Each representation reveals a different way of thinking about the problem. Giving attention to the different methods as well as to the different representations will help students see the power of viewing a problem from different perspectives. Notice that the teacher added a different representation to provide another way to view the problem. Modeling ways to organize the information and drawing attention, through discussion, to the value of organized representations is necessary for many students.

The language of mathematics includes techniques for recording and organizing ideas that are unique to the discipline. Students learn many symbols (numerals, operation symbols, an equals sign), conventions (e.g., number sentences) and organizational schemes (e.g., labeled charts) in the primary grades. In grades 3–5, their repertoire of symbols, tools, and conventional notation should expand and be clearly connected to concepts as they are explored. Students in these grades should acquire the ability to set up a simple spreadsheet (figure 5.27), to write a mathematical equation and to draw and label diagrams accurately. Their representations should be more detailed and useful than is expected in the primary grades. For example, the recipe spreadsheet includes the capability to go from a single batch to a batch-and-a-half or a double batch.



	A	B	C	D
1			COOKIE RECIPE	
2				
3			Amounts Needed	
4	Ingredients	for 24 cookies	for 36 cookies	for 48 cookies
5	Flour (cups)	1		
6	Granulated sugar (cups)	0.5		
7	Brown sugar (cups)	0.5		
8	Baking soda (tsp)	0.4		
9	Cinnamon (tsp)	0.5		
10	Quick rolled oats (cups)	1.5		
11	Butter (cups)	0.5		
12	Eggs	1		
13				
14				

Figure 5.27. Cookie spreadsheet.

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Learning to interpret, use, and construct useful representations needs careful and deliberate attention in the classroom (Greeno and Hall 1997). Teaching forms of representation (e.g., graphs, charts, number sentences) as ends in themselves is not productive. Rather, they are useful tools for building understanding and for communicating both information and understanding. They acknowledge that both standard and nonstandard and physical and mental representations serve a purpose and that teachers should encourage flexibility in choosing and creating representations that fit the purpose at hand. Finally, they encourage teachers to involve students in discussions about the advantages and limitations of a particular representation.

Chapter 6
Standards for Grades 6–8

The goal of mathematics instruction in the middle grades is to lay a solid mathematical foundation for secondary school and beyond. During this time period, many students will form conceptions about themselves as learners of mathematics—about their interest, their competence, their attitude, and their motivation. These conceptions will influence how they approach the study of mathematics in high school, which will in turn influence their life opportunities.

A strong foundation will be laid if worthwhile mathematics is taught and if it is taught well. In the remainder of this chapter, key instructional emphases are identified for grades 6–8 for each of the ten standards identified in chapter 3. Instructional programs that focus on these areas of emphasis will help students build a strong foundation for further study of mathematics and for the use of mathematics in dealing with quantitative matters throughout their lives. For each of the five content standards, elaborations specific to grades 6–8 are provided, organized around the key focus areas. In contrast, the elaborations for the five process standards are organized differently, using questions that call attention to the nature of these processes in grades 6–8 and the teacher's role in fostering students' learning with respect to them. These two organizational schemes offer different glimpses into the nature of a strong mathematics instructional program in grades 6–8, attending both to mathematics content and process.

Teaching Mathematics to Young Adolescents

In addition to careful consideration of the content of the mathematics instructional program in grades 6–8, attention also needs to be paid to the settings in which students learn mathematics. It is critical for students in the middle grades to find both challenge and support within the mathematics classroom. High expectations and strong support must go hand in hand. Given the sensitivity of young adolescents to their peers, it is particularly important for their teachers to establish classroom learning communities where the norms support student learning of mathematics. And the middle grades mathematics classroom should be a place in which young adolescents regularly engage in thoughtful activity that relates to their emerging capabilities of finding and imposing structure, conjecturing and verifying, and engaging in abstraction and generalization. Middle grades students should feel drawn into mathematics rather than driven away from it.

Teaching for Thoughtful Engagement and Learning With Understanding

As asserted in the Teaching and Learning Principles in chapter 2, mathematics instructional programs should aim for teaching and learning with understanding and they should engage students in thoughtful activity in the classroom. To lay a solid foundation for secondary school and beyond, students in grades 6–8 need to develop a sound understanding of many mathematical ideas, including ratio, proportion, percent, equality (as equivalence), variable, area, volume, similarity, and symmetry. Toward this end, students need regular experience with interesting, challenging tasks through which they develop meaning for mathematical ideas. Although this may seem obvious, there is evidence (e.g., Stigler and Hiebert 1997) that meaning and challenge are not consistently emphasized in middle grades mathematics classrooms. Too often, mathematical ideas are only stated, rather than developed in ways that support understanding. Moreover, the tasks given to students are too often routine, and their completion relies more on memory and imitation than on thinking and reasoning.

Students need regular experience with interesting tasks that engage them in mathematical thinking, reasoning, and problem solving. Although teachers face challenges when they try to use cognitively demanding tasks in the mathematics classroom (Stein, Grover, and Henningsen 1996), evidence suggests that students benefit from mathematics instruction that includes cognitive challenge and that attends to the development of meaning (Knapp et al. 1995; Stein and Lane 1996). Intellectual engagement and struggle contribute to learning. Moreover, young adolescents can derive personal satisfaction as a consequence of grappling with interesting and challenging problems. As the elaborations in this chapter suggest, this kind of thoughtful approach to mathematics teaching and learning can be developed in grades 6–8 through experiences in all the areas addressed by the standards in this draft.

Breadth and Focus in the Curriculum

As in the *Curriculum and Evaluation Standards* (NCTM 1989), a broad curriculum is proposed here for the middle grades. As we approach the new millennium, it is as important to emphasize a need for breadth in the mathematics curriculum for grades 6–8 as it was in 1989. Therefore, important teaching and learning goals are identified for grades 6–8 in each of the five content strands, and the importance of each of the five process standards is also discussed.

For some readers, an emphasis on breadth may appear to imply a lack of focus. The curriculum in the United States has been criticized for being "a mile wide and an inch deep" (Schmidt, McKnight, and Raizen 1996, p. 34). But breadth and focus are not mutually exclusive; both are critical features of a quality mathematics program for the middle grades. Within a program that treats a broad range of mathematical content and processes, it is also possible to have clear focus areas for instruction and learning, such as the development of fluency with rational number computation and facility with proportional reasoning.

Developing Fluency with Rational Numbers

Number is an important component of the mathematics program in the middle grades, but the instructional emphasis needs to be on rational numbers, not whole numbers. In some schools and classrooms, too much attention is given to teaching whole-number computation in grades 6–8. The clear focus proposed here for grades 6–8 with respect to number is the development of fluency with rational number calculation, with the expectation that instruction in whole-number computation will be completed by the end of grade 5. Some attention to number theory and very large numbers will be part of grades 6–8. Nevertheless, it is expected that work with whole numbers will occur naturally and often in the context of problems, especially those involving data and measurement.

Major instructional attention in the middle grades should be directed toward helping students develop a deep understanding of rational number concepts and reasonable proficiency in rational-number computation. As a consequence, students should be able to think flexibly about relationships among fractions, decimals, and percents. Students' understandings of, and proficiency with, rational numbers can be developed naturally and often in the context of problems that involve geometry, measurement, data, probability, patterns, functions, and algebra. Moreover, their work with rational numbers in grades 6–8 will also support their development of proportional reasoning.

Building Facility with Proportional Reasoning and Linearity

One of the hallmarks of the middle grades mathematics program should be the development of facility with proportional reasoning. Proportional reasoning permeates the entire middle grades curriculum, is a key integrative thread that connects most of the mathematics topics studied in the middle grades, and can be developed in several areas of mathematics. Proportional reasoning involves much more than setting two ratios equal and solving for a missing term. Critical development points in grades 6–8 include work with ratios and proportions, percent, similarity, scaling, linear equations, area, volume, data distributions, and probability.

Students encounter proportional relationships when they study linear functions of the form $y = kx$, when they consider the relationship between points on a map drawn to scale and the actual distance between the corresponding locations in the world or when they estimate the number of defective objects in a population of manufactured goods by inferring from an inspection of a sample drawn at random from the population. Proportional reasoning is involved in considering the relationship between the circumference of a circle and the length of its diameter or the length of the diagonal of a square in relation to the length of its side. Topics as apparently diverse as sales tax, slope, and similarity can all be seen to be related when thought about as instances of proportional relationships (i.e., between the tax levied and the base cost of items, between vertical and horizontal distances traveled between points, and between the lengths of corresponding sides of similar figures, respectively). In order for students to see the connections among these diverse experiences and topics, they must engage in reasoning about the underlying proportional relationships.

Of course, not every mathematical relationship encountered in the middle grades is a direct, linear proportion. Some relationships involve inverse variation (e.g., the relationship between length and width in a rectangle with fixed area), some involve nonlinear variation (e.g., the relation between the volume of a cube and the volume of the cube after its sides have all been multiplied by the same factor), and others have both additive and proportional components (e.g., the relationship between the Fahrenheit and Celsius expressions of a given temperature). Nevertheless, linear proportional relations appear often, both within and outside mathematics, and students in the middle grades need to learn to recognize such relationships and to work with them effectively.

Proportional reasoning is also closely tied to the development of fluency with basic algebraic concepts and skills in the middle grades. Many topics studied in the middle grades involve linear proportional relations, which can be stated in the form $y = mx$. Experiences with these topics can help students build a strong understanding of important algebraic ideas and skills, such as linear functions and slopes of lines. An understanding of linearity in the middle grades can lay a solid foundation for the study of linear algebra, linear approximation, and nonlinear functions in secondary school and beyond. And as the above discussion of proportional relationships and reasoning suggests, these rich understandings of algebraic ideas can be developed in the middle grades in a way that connects these algebraic understandings to many other content areas and topics.

Fostering Algebraic Competence Without Acceleration or Specialization

In recent years in the middle grades there has been a trend toward acceleration and specialization related to algebra. Students in grade 8, or even in grade 7, are increasingly taking courses that focus exclusively on algebra and that are organized in a manner similar to the traditional algebra course in high school. This trend represents a well-intentioned response to what many perceived to be a lack of quality and challenge in the mathematics program for middle grades students. However, such acceleration and specialization can have negative consequences for students. One negative consequence is that students are likely to have less opportunity to learn the full range of mathematics content, especially topics in geometry and data analysis, that are expected in the middle grades.

A strong mathematics program for the middle grades does not require acceleration and specialization. Students should have substantial experience with algebraic ideas in the middle grades, but this does not require that students take a course that focuses exclusively on this topic, especially if that course does not engage students in thoughtful activity aimed at building understanding (Steen 1992). In fact, there is considerable value in connecting the learning of algebraic concepts and skills to the study of other mathematical content topics. Proportional reasoning can be a key element in this integrated development.

Meeting These Standards Requires Long-term Commitment and Support

This draft of *Principles and Standards* offers an ambitious agenda for middle grades mathematics education. Students are expected to learn serious, substantive mathematics in classrooms where the emphasis is on thoughtful engagement and meaningful learning. It will be insufficient simply to announce this as the new standard for middle grades mathematics education; it will also be necessary to provide long-term commitment and support to this agenda. The capacity of schools and teachers to provide this kind of mathematics education needs to be developed. Special attention must be given to the preparation and ongoing professional support of teachers in the middle grades to help them develop knowledge of mathematical ideas, learn excellent pedagogical practices, and become familiar with research on students' mathematics learning.

#8

Reader Reaction

As you read the next sections, please consider the following:

We are interested in your reaction to the style and tone of Chapter 6. How might the chapter be more helpful?

What is your reaction to the perspective on algebra that is presented here?

Do the set of Standards for 6–8 adequately describe a rich middle grades mathematics program? Is anything missing? Please specify.

Standard 1: Number and Operation

Mathematics instructional programs should foster the development of number and operation sense so that all students—

- **understand numbers, ways of representing numbers, relationships among numbers, and number systems;**
- **understand the meaning of operations and how they relate to each other;**
- **use computational tools and strategies fluently and estimate appropriately.**

Elaboration: Grades 6–8

Rational numbers are the central focus of middle school students' experiences with number, and they play a critical role in the development of facility with proportional reasoning. In grades 6–8, students build on experiences with fractions, decimals and percents in earlier grades or in everyday life, and they also bring knowledge of whole number concepts and fluency with operations. This provides a base on which to build a solid understanding of rational numbers. In general, students' work with whole numbers will come in problem-solving contexts across the curriculum, with only a few whole number topics receiving specific instructional attention in the middle grades. In the middle grades, students also learn about integers and are introduced to a few irrational numbers.

By the time they leave middle school, students should have accurate and efficient strategies for computing with fractions, decimals, percents, and integers, and they should be able to use numerical estimates accurately and appropriately. This goal is likely to be reached only if substantial attention is also given to developing a solid conceptual understanding. Research suggests that part of the reason for students' difficulties with rational number computation is that they often learn to manipulate symbols without meaning (Hiebert and Behr 1988). There are numerous conceptual issues with which students need to grapple in coming to understand rational numbers. For example, the differences and similarities among the meanings and uses of ratios, rates, and fractions are not at all trivial. Moreover, multiplying or dividing fractions and integers requires a different conceptualization of these operations than was the case for whole numbers. Students need many opportunities to think deeply about rational numbers and to use them to solve problems. Experience with rational numbers occurs naturally in the study of many topics in the curriculum. For example, students can use ratios, rates, fractions, decimals, or percents to give measurements, to express probabilities or ratios of similarity, to compare survey responses of two unequal sized samples, or to represent slope or rate of change in a linear equation or problem setting.

Focus Areas for Grades 6–8

- **Understand numbers, ways of representing numbers, relationships among numbers, and number systems**

  In grades 6–8, all students should—

  - work flexibly with equivalent fractions, decimals, percents; compare and order these numbers efficiently and accurately, find their approximate locations on a number line; and choose appropriate and convenient forms of these numbers for solving problems;
  - develop meaning for integers and be able to represent, compare, and order them;
  - develop an understanding of large numbers, including the use of benchmarks to comprehend their magnitude; and recognize, understand and appropriately use various representations for large numbers (e.g., exponential, scientific, and calculator notation);
  - use number theory concepts (e.g., factors, multiples, prime factorization, relatively prime numbers) to solve problems and to understand ideas about rational numbers;
  - develop an understanding of the properties of the integer and rational number systems (e.g., order, density, and additive and multiplicative inverses);
  - recognize and use commonly encountered irrational numbers (e.g., π and $\sqrt{2}$).

- **Understand the meaning of operations and how they relate to each other**

  In grades 6–8, all students should—

  - extend understanding of operations to include operations on fractions, decimals, percents, integers, and nonnegative whole number exponents;
  - understand the effects of operating among fractions, decimals, percents, and integers;

- recognize and use the properties of operations on integers and other rationals, such as closure, associative, commutative, and distributive properties;
- understand and use the inverse relationships of addition and subtraction, multiplication and division, and square and square roots to solve problems;
- extend understanding of counting to include elementary combinatorics.

- **Use computational tools and strategies fluently and estimate appropriately**

  In grades 6–8, all students should—

  - develop, analyze, and compare algorithms for computing with fractions, decimals, percents and integers, and become efficient and accurate in computing with them;
  - develop, analyze and explain methods for solving problems involving proportions (e.g. scaling, finding equivalent ratios);
  - develop and refine strategies for estimating (including fractional quantities); use estimations as a means to check the reasonableness of results;
  - select and use appropriate methods for computing from among mental arithmetic, estimation, paper-and pencil, and calculator, depending upon the situation at hand.

Discussion

- **Number Concepts and Their Properties**

Students in the middle grades recognize a need for numbers other than whole numbers, and they enter these grades with some familiarity with fractions, decimals, percents, and integers. Developing deep understanding, fluency, and flexibility in working with these numbers is an important focus of the middle grades. Students need to recognize many meanings and uses of fractions—representing not only parts of a whole, but also indicated divisions (e.g., $3/4 = 3 \div 4$), measures or quantities, decimals, percents, rates, and ratios. Decimal numbers need to be understood not only as fractions whose denominators are powers of 10, but also as a natural extension of the base-ten, place-value system for representing quantities less than 1.

Students should learn to identify equivalent representations of numbers and to translate freely among them, recognizing the advantages and disadvantages of each representation as tools for understanding and problem solving. Although 15/100, 3/20, 0.15, and 15% are all representations of the same number, it is usually more useful to write a fraction of a dollar as 15/100, the probability of winning a game as 3/20, the tax on a purchase of \$2.98 as \$0.15, and a sales discount as 15%. Students also should come to recognize the difficulties inherent in certain representations, such as writing 1/3 as a percent or seeing 5.99999999 on a calculator instead of 6. To develop such understanding, students need thought-provoking rather than mechanical experiences. Students need to solve problems and investigate situations designed to foster flexible thinking about rational number representations. Experience with a variety of models for rational numbers—fraction strips, number lines, 10-by-10 grids, area models, and discrete sets of objects—can offer students concrete representations for abstract ideas and support their thinking and problem solving. Also, experience with concrete models offers further opportunity for students to compare representations and move flexibly among them to solve problems.

Middle grades students also work with integers. Typically they already have intuitive understandings of these numbers from experiences in their everyday lives: temperatures in winter that drop below zero, games such as football where teams gain or lose yards on each play, news reports regarding stock market fluctuations, or their own speculation about what happens when 7 is subtracted from 5. The school curriculum needs to build and expand on these informal experiences. For example, to help students understand that –12 is less than –5, it may be useful to refer to temperatures and their location on a thermometer, which then leads to extending the number line to include negative numbers.

Although the focus in the middle grades is on rational numbers, whole numbers are treated when students work with very large numbers (millions and billions). They need to understand the magnitude of these numbers and to become facile in reading and representing such numbers (e.g., 2,300,000,000 can also be represented as 2.3 billion or as 2.3 x 10^9 in scientific notation). The use of familiar quantities (e.g., the size of a football field, or the population of one's state or province) as benchmarks can often help students understand these large magnitudes. Contexts encountered in other school subjects as well as in everyday life can serve as problem situations for the study of large numbers. For example, students see newspaper headlines such as "Clean-Up Costs from Oil Spill Exceed $2 Billion!", and they see numbers such as 2.5 x 10^{11} in science textbooks to indicate the number of red blood cells in the human body. Students need to be able to interpret numbers of this magnitude and to understand various forms of notation (e.g., the number 2.5 x 10^{11} might appear on a calculator as "2.5E11" or "2.5 11", depending on the make and model of the machine).

Students also work with whole numbers in their study of number theory concepts. Problems concerning divisibility (e.g., show that the product of any four consecutive integers is always divisible by 24), prime numbers (e.g., show that the sum of two primes is not necessarily prime), or figurate numbers (find the 20th triangular number) not only build on students' knowledge of whole numbers but also provide rich opportunities for problem solving, reasoning, and the use of algebraic thinking. Moreover, students' learning about prime factorization and relatively prime numbers helps them understand some aspects of their work with fractions (e.g., procedures for simplifying or adding fractions).

The middle grades curriculum should also develop students' ability to make intelligent comparisons of quantitative information by using ratios and rates. Formulating ratios in order to make comparisons is at the heart of proportional reasoning, a key concept in many areas of mathematics (e.g., similarity in geometry, patterns of growth in algebra). Developing students' understanding and strategies for making appropriate comparisons takes time and a variety of experiences. For example, when trying to decide which is the better buy—12 tickets for $15 or 20 tickets for $23—students might choose from a number of different strategies. They might use scaling to find how much 60 tickets would cost in each case (12 tickets for $15 means 60 tickets for $75, whereas 20 for $23 means 60 for $69—the better buy). Alternatively, unit rates could be used to determine the better buy ($15 for 12 means $1.25 for 1, whereas $23 for 20 means $1.15 for 1). The focus for instruction is not only to have students experience a variety of different ways to reason in proportional situations, but also to help them learn to recognize when proportional reasoning is appropriate.

• Operations and Their Properties

Students should come to the middle grades with a solid understanding of whole number operations. Experiences at this level continue to develop students' understandings of operations as they compute with fractions, decimals, percents, and integers. Understanding an operation includes being able to reason about what its effect will be on the numbers to which it is applied, and knowing whether the operation is actually appropriate to solve a particular problem.

Students' ideas of addition, subtraction, multiplication, and division expand as they learn to operate with rational numbers. For example, students' understanding of whole-number addition and subtraction can be expanded to fractions and decimals if they have a strong understanding of the meaning of numerators and denominators, size of the number, equivalence, and place value. Understanding place value allows students to compute 1.4 + 0.67 by using what they know about adding 140 + 67. Similarly, students with strong fraction sense and a firm understanding of addition are unlikely to commit the classic error of indicating that 2/3 + 3/4 is equal to 5/7. This answer is not reasonable because each of the addends is more than 1/2, so the sum must be greater than 1. A firm understanding of the relationship between the operations of addition and subtraction, together with conceptual understanding about distances on a number line, makes developing a conceptual understanding of adding and subtracting integers easier.

Multiplying and dividing fractions and decimals challenge ideas that many students have developed about these operations from their work with whole numbers. Often students believe that multiplication always makes things bigger and that division always results in a smaller number for an answer. These ideas can be troublesome for making sense of multiplication and division of fractions (Graeber and Tanenhaus 1993). But if students have already thought about multiplication of whole numbers not only as repeated addition, but also by using an area model; then they can use the area model as an aid to understanding multiplication of fractions or decimals. Similarly, if students have thought about division of whole numbers not only as sharing (if 24 cookies are shared among 6 friends, how many cookies does each friend get?) but also as repeated subtraction (if packing 24 cookies into packages containing 6 cookies each, how many packages could you fill by packing 6 cookies at a time?), then they can use the latter model to think about division of fractions. For example, if 4 1/2 yards of ribbon needs to be cut into pieces that are 3/4 yard each for bows, how many bows can be made? The image is of repeatedly cutting off 3/4 of a yard of ribbon. Having students work with concrete objects or drawings is helpful as students develop and deepen their understanding of operations. Students need to regularly revisit the meaning of operations as they work with rational numbers and integers in the middle grades. Consideration of division by zero is appropriate for discussion at this level.

- **Computation and Estimation**

Students are expected to bring to the middle grades a firm understanding of whole-number operations and efficient procedures for operating with these numbers, so valuable instructional time in the middle grades should not be spent on whole-number computation. Nevertheless, students should have many opportunities to compute with whole numbers as a natural part of solving problems encountered in the middle grades curriculum, such as calculating area or volume, determining the mean or interquartile range of a numerical data set, or generating successive terms of a sequence.

In grades 6–8, students acquire fluency in adding, subtracting, multiplying, and dividing with fractions, decimals, and integers. The development of algorithms for operating with these numbers should be based on student work with carefully selected tasks, accompanied by discussions that are orchestrated by the teacher. By talking about problems in context, students can develop meaningful computational algorithms. Sharing algorithms with each other, explaining why algorithms work and make sense, and comparing student-developed algorithms to the algorithms traditionally taught in school helps students identify reasonable methods for computing with fractions, decimals, and integers. The development of computational procedures for calculations involving percents should follow a similar path. Instruction regarding the solving of proportions should give attention to methods that have a strong intuitive basis, such as scaling and finding equivalent ratios. The "cross-multiplication" method is less intuitive, and students may not be able to understand the reasonableness of the method until they can use algebraic symbolism to justify it.

Computation, in this draft, as was also the case in the *Curriculum and Evaluation Standards* (1989), involves a broadening of the traditional view to include not only paper-and-pencil computation but also mental computation and the use of calculators and other calculation tools. Because of this expanded definition, computation must be tightly linked to reasoning and estimating. In any particular situation, students need to decide whether an exact answer or an estimate is appropriate, and also whether it makes sense to solve the problem mentally, with paper and pencil, or with a calculator. For example, the cost of 1 1/4 pounds of nuts at $2.40 per pound could be found mentally, whereas the cost of 1.37 pounds of bulk candy at $1.17 a pound might be estimated, although a calculator would probably be the preferred tool if an exact answer were needed. Students also should be able to do this computation with paper and pencil, if necessary. Helping students decide when exact answers or estimates are appropriate, which computational method to use, and how to judge the reasonableness of answers are part of the teaching and learning.

Standard 2: Patterns, Functions, and Algebra

Mathematics instructional programs should include attention to patterns, functions, symbols, and models so that all students—

- **understand various types of patterns and functional relationships;**
- **use symbolic forms to represent and analyze mathematical situations and structures;**
- **use mathematical models and analyze change in both real and abstract contexts.**

Elaboration: Grades 6–8

Algebra can be thought of in many ways—as patterns, functions, and relations; as language, representations, and structures based on generalized arithmetic; or as a tool for modeling mathematical ideas and problems. All of these views of algebra are found in the middle grades. Although they are discussed here separately, they would be interwoven in a quality school mathematics instructional program.

The study of algebra in the middle grades involves an intermingling of conceptual and procedural work. Students work more formally with representations and symbolism than they did in their earlier school experience, but their instruction should have as its primary emphasis the development of a sound conceptual foundation. It is essential that students become proficient in working with simple algebraic expressions, including generating equivalent expressions and using and evaluating simple formulas. By the end of the middle grades, students should be able to solve linear equations in several ways, with or without a calculator, as appropriate. But it is also critical that they develop a conceptual understanding of variables and equations, that they learn to distinguish linear relationships from nonlinear ones, and that they connect their experiences with linear functions to their understandings of ratio and proportion. Through their experiences in the middle grades, students should develop an understanding of, and competence in, working with linearity.

Focus Areas for Grades 6–8

- **Understand various types of patterns and functional relationships**

  In grades 6–8, all students should—

  - analyze, create, and generalize numeric and visual patterns paying particular attention to patterns that have a recursive nature;
  - use patterns to solve mathematical and applied problems;
  - represent a variety of relations and functions with tables, graphs, verbal rules, and, when possible, symbolic rules.

- **Use symbolic forms to represent and analyze mathematical situations and structures**

  In grades 6–8, all students should—

  - develop a sound conceptual understanding of equation and of variable;
  - explore relationships between symbolic expressions and graphs, paying particular attention to the horizontal and vertical intercepts, points of intersection, and slope (for linear relations);
  - become fluent in generating equivalent expressions for simple algebraic expressions and in solving linear equations and inequalities;
  - use symbolic algebra to represent situations and to solve problems, especially those that involve linear relationships.

- **Use mathematical models and analyze change in both real and abstract contexts**

  In grades 6–8, all students should—

  - model and solve contextualized problems using various representations, such as graphs and tables, to understand the purpose and utility of each representation;
  - develop an initial understanding of rate of change, with emphasis on the connections among slope of a line, constant rate of change, and their meaning in context;
  - explore different types of change occurring in discrete patterns, such as proportional and linear change.

Discussion

• Patterns and Functions

A major goal in the middle grades is to develop students' facility with analyzing, modeling, representing, and solving mathematical and real-world problems using patterns. Recursive patterns, that is, patterns in which each number is found from the previous numbers by repeating some process (e.g., figurate numbers and Fibonacci numbers), are becoming increasingly useful in these situations. The following example shows how recursive patterns and the use of a variety of different representations can aid in understanding and solving a problem related to geometry. Note how each representation adds a different dimension to the understanding. Specifically observe how the table of values allows students to see patterns, and how the figures help students understand why the patterns occur. An important learning goal is for students to develop the disposition of looking for reasons why patterns occur.

Suppose students are studying the relationship between the number of sides of a polygon and the sum of the interior angles or the total number of diagonals that can be drawn. They might organize their information as in table 6.1.

Number of sides	3	4	5	6
Sum of interior angles	180	360	540	720
Number of diagonals	0	2	5	9

Table 6.1. Angle sums and diagonals for various polygons.

If studying interior angles, some students might observe that as the number of sides increases by one, the sum of the interior angles increases by 180. If studying diagonals, they may observe that the differences of consecutive pairs of numbers in the sequence 0, 2, 5, 9 is 2, 3, and 4. They then might conjecture that the numbers of diagonals in seven- and eight-sided polygons are $9 + 5 = 14$ and $14 + 6 = 20$, respectively. Other students might observe that the number of diagonals increases by $n - 2$, where n is the number of sides, so the numbers of diagonals of a seven- and eight-sided polygon is $9 + (7 - 2) = 14$ and $14 + (8 - 2) = 20$, respectively. Other students may observe that the number of degrees in the sum (S) of the interior angles of any polygon with n sides seems to be given by the equation $S = 180(n - 2)$, which happens to be a linear function.

Middle grades students should be encouraged to support their conjectures not only by appealing to the patterns, but also by logical arguments that explain why the generalizations must be true. By exploring with drawings or geometry software, students could see that when the number of sides of a polygon is increased by one, the sum of the interior angles increases by 180 degrees because another triangle is added. This is seen in the left polygon of figure 6.1 in which a four-sided polygon is converted to a five-sided polygon. Students could justify the formula $S = 180(n - 2)$ by drawing all the diagonals from any single vertex to form precisely $n - 2$ triangles.

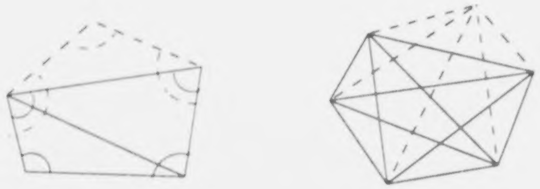


Figure 6.1. Diagonals of a quadrilateral and a pentagon.

Students could justify that the number of diagonals increases by $n - 2$ by seeing that when another vertex is added, as in the right polygon of figure 6.1 (a five-sided polygon is converted to a six-sided polygon), $n - 3$ diagonals are added to the new vertex, and one side becomes a diagonal. If students have solved the well-known "handshake" problem, in which each person in a group shakes hands with each other person, they may be able to use similar logic to conclude that the total number of diagonals should satisfy $T = n(n - 3)/2$.

Recursive approaches, in which students reflect on how the numbers are changing, and closed-form solutions, such as $S = 180(n - 2)$, are legitimate methods for solving problems and both should be valued. When using a recursive approach, students should clearly write the relationship, such as "each number is 180 more than the previous number" or "next = 180 + previous". An even more powerful approach is to have students use spreadsheets to generate the numbers in the sequence. Because of technology, it is just as easy to generate almost any number needed using the recursive approach as it is using the formula. Because of this, recursive thinking is an important skill that will be built upon in high school.

- **Structure and Symbolic Notation**

It is important to realize that developing students' understanding of variable is complicated, precisely because the concept of variable itself is multifaceted (Usiskin 1988). A deep understanding of variable involves much more than simply recognizing that letters can be used to stand for unknown numbers in equations. Consider these expressions:

$$30 = 5x \qquad 1 = n(1/n) \qquad A = LW \quad y = 3x.$$

In the first equation, x stands for a specific, nonvarying number that can be determined by solving the equation. The second equation generalizes an arithmetic pattern where n can take on a wide range of values (but not zero); it represents an identity. The third equation is usually considered a formula with A, L, and W standing for the quantities area, length, and width. These quantities actually feel more like knowns than unknowns. The last equation offers a more compelling example of "variability" than any of the others, since x takes on a variety of values, and the corresponding values for y can be computed. A graph could be used to represent the equation.

Middle grades students need a wide variety of experiences with equations of various sorts before they can be expected to develop a deep understanding of variable and equation. Although a complete understanding will probably not be solidified during the middle grades, by the end of grade 8 students should be able to solve equations like $30 = 5x$ for the unknown number, to recognize equations such as $1= n(1/n)$ as identities (when n is not zero), to use formulas such as $A = LW$ in applied situations, and to recognize $y = 3x$ as a linear equation that holds true for many ordered pairs (x,y).

Students should become fluent with the graphical representation and solution of linear equations and simultaneous systems of linear equations. As with skill in other forms of symbolic manipulation, it is also important that this fluency be based on a sound conceptual foundation to ensure that these manipulations are meaningful and not merely rote exercises. In particular, middle grade students should understand and use the facts that 1) ordered pairs that satisfy two linear equations are precisely the set of points that lie simultaneously on the graphs of those equations; 2) the x- and y-intercepts of the graph of any function $y = f(x)$ correspond to places where y and x, respectively, are zero; and 3) for linear functions $y = mx + b$, the slope can be thought of as the steepness of the graph, how fast the function is rising or falling, or as a rate of change connected to a physical concept such as speed.

Learning to interpret and write symbolic expressions and to solve symbolic algebraic equations is important in the middle grades, but for most students extensive work with symbols should come only after they have become fluent with verbal, tabular, and graphical representations of relationships that symbols represent. Many teachers find it helpful to introduce algebraic expressions as models of quantities in contextual situations, such as the following. Suppose three friends attend a ball game and each purchases a hot dog and a soda. The total cost could be denoted $3\ (H + S)$, where H is the cost of a hot dog and S is the cost of a soda. But it is also possible to think separately about the cost of the hot dogs ($3H$) and the cost of the sodas ($3S$). In this case, the total cost would be denoted $3H + 3S$.

By working on problems like this, students can gain experience with symbolic algebraic representations of situations. Moreover, they can see that different symbolic expressions can be produced to represent the same situation. If these expressions are all valid, then the expressions should be equivalent. This can lead to an examination of algebraic equivalence. Also through such experiences, students learn that they can often examine the equivalence of expressions in many ways, such as by checking that the related graphs and tables are the same, or by using the distributive, associative, and commutative properties to transform expressions into equivalent forms. By the end of grade 8, most students should be able to use their understanding of equivalence and of the commutative, associative, and distributive properties to recognize the nonequivalence of expressions such as $\sqrt{36+9}$ and $\sqrt{36} + \sqrt{9}$, and the equivalence of expressions such as $2(x + 3)$ and $2x + 6$. Through such experiences, students can give meaning to work with algebraic expressions.

• Modeling and Change

Students learn to think about algebra as a valuable tool for modeling when they create equations or graphs to represent contextualized situations. But the reverse is also important—that is, for students to be asked to interpret graphs or equations as models of situations. For example, students might be shown a piecewise linear graph of distance from home versus time, or total cost of tickets versus number of tickets bought, and asked to describe a situation that the graph might model. Teachers need to encourage students to provide clear, complete oral and written descriptions for such situation models. Asking interpretive questions can also help: When was the person at home? How much did the tickets cost? To answer these questions, students must be able to interpret intercepts and slope in a graph.

In the middle grades, students should have many experiences exploring problems using multiple representations, such as in investigating interior angles and diagonals of polygons, as discussed earlier. Through work with patterns, graphs, and symbols, middle grades students can consider change and co-variation — how changes in one variable affect another.

It is particularly important for middle grades students to gain a deep understanding of rate of change and how it relates to lines and slopes of lines. Students may have difficulty understanding rate of change, since it is often a derived object, such as velocity, and not a physical entity. Teachers can help overcome this problem by giving students experiences with problem situations in which change is occurring and by helping students reflect on the connections among the actual change, its visual representation on a graph, and the constant m in linear equations $y = mx + b$. Such experiences could relate to experiences with technology-based laboratories, computer simulations, or patterns such as $S = 180n - 360$, the relationship between the number of sides of a polygon and the sum of the interior angles. Students should also understand that any change must take place with respect to something: change in position over time, change in size over time, change in area with respect to change in sides, and—for patterns of numbers—change with respect to position in the sequence.

Middle grades students should also explore other functions, such as quadratics and exponentials. The derivation of the formula $T = n(n - 3)/2$ as the number of diagonals in an n-sided polygon, allows students to see quadratics arising naturally. By reflecting on the recursive approach to this problem, that is, that the *change* in the number of diagonals is increasing, 0, 2, 5, 9, students begin to understand the nature of the increasing and decreasing "slope" of quadratics. Exponential functions may also arise naturally in the middle grades when students consider situations such as this one: Put one grain of wheat on the first square of a chessboard, two grains on the second square, four grains on the third, and so on. In reflecting on the fact that the change is doubling, or is a ratio, students begin understanding the shape of exponential functions. Students should be able to recognize graphs of each type of relation, and they should be able to recognize the way in which these relations manifest themselves in tables of data. Another important goal is to assist students in building a solid understanding of the interplay among tables of data, graphs, and algebraic expressions as models of the relations and functions that arise from these problem situations. Beginning the development of an understanding of change is central to the study of functions and much of calculus in secondary school and beyond.

Standard 3: Geometry and Spatial Sense

Mathematics instructional programs should include attention to geometry and spatial sense so that all students—

- **analyze characteristics and properties of two- and three-dimensional geometric objects;**
- **select and use different representational systems, including coordinate geometry and graph theory;**
- **recognize the usefulness of transformations and symmetry in analyzing mathematical situations;**
- **use visualization and spatial reasoning to solve problems both within and outside of mathematics.**

Elaboration: Grades 6–8

The study of geometry in grades 6–8 builds on the informal explorations in grades pre-K–5 and lays a foundation for the further study of geometry in high school. Geometry in the middle grades engages students in investigating geometric relationships by constructing, drawing, measuring, visualizing, comparing, transforming, and classifying geometric figures. This work both builds upon and further develops students' spatial sense.

Geometry in the middle grades also provides a rich context for the development of students' mathematical reasoning, including both inductive and deductive reasoning, making and validating conjectures, and classifying and defining geometric objects. Students' conjectures, verifications, and associated discussions lead to the specification of relationships among two-dimensional and three-dimensional figures, and they help lay the groundwork for understanding both the need for and the nature of mathematical definitions. Inductive inferences and deductive conclusions generated by students in the middle grades presage the more formal validations using proof in which students may engage in high school.

In the middle grades, the study of geometry also provides representational tools for understanding other topics in mathematics. For example, coordinate geometry is used to represent algebraic expressions, and geometric representations can be used for algebraic and numerical relationships. Moreover, many topics treated in the measurement standard for the middle grades, such as area and volume, are intimately tied to students' study of geometry.

Focus Areas for Grades 6–8

- **Analyze characteristics and properties of two- and three-dimensional geometric objects**

  In grades 6–8, all students should—

  - precisely describe, classify, and compare types of plane and solid figures (e.g. angles, triangles, quadrilaterals, cylinders, cones, etc.) according to their main features;
  - analyze and understand geometric relationships among two-dimensional and three-dimensional figures;
  - use proportions to examine relationships between similar plane figures;
  - create and critique inductive and deductive arguments concerning geometric ideas and relationships;
  - recognize and apply geometric ideas and relationships outside the mathematics classroom, in areas such as art, science, and everyday life.

- **Select and use different representational systems, including coordinate geometry and graph theory**

  In grades 6–8, all students should—

  - learn to use coordinate geometry to display geometric relationships between related quantities, particularly when the relationship is linear;
  - use coordinate geometry to represent and examine the properties of geometric figures, especially those containg sets of parallel or perpendicular lines;
  - use relationships found in right triangles (e.g. the Pythagorean relation, isosceles right triangles, 30-60-90 degree triangles) to solve problems;
  - explore the use of other representational systems, particularly networks.

- **Recognize the usefulness of transformations and symmetry in analyzing mathematical situations**

  In grades 6–8, all students should:

  - describe size, position, and orientation of figures under informal transformations such as flips, turns, slides, and magnification;
  - use line and rotational symmetry to describe and classify polygons and polyhedra;
  - understand the concepts of congruence and similarity using transformations;
  - explore the composition of transformations (e.g., successive flips in different lines).

- **Use visualization and spatial reasoning to solve problems both within and outside of mathematics**

  In grades 6–8, all students should:

  - develop fluency with two-dimensional representations of three-dimensional objects;
  - compose and decompose two- and three-dimensional figures in order to solve problems;
  - use geometric models to represent and explain numerical and algebraic relationships.

Discussion

• Shape and Structure

Students come to the study of geometry in the middle grades with a great deal of relevant experience. Geometric figures surround students in their daily lives, and they can derive insights about the geometric representation of figures from this experience. They also arrive with knowledge about the concepts of point, line, plane, and a variety of shapes; with much experience in visualizing and drawing lines, angles, triangles, and other polygons; and having worked with and developed descriptions of various two- and three-dimensional figures. In the middle grades, students should become more precise about the defining features of different types of polygons and polyhedra and use these to classify, compare, and contrast.

Much of geometry in the middle grades is concerned with the exploration and precise specification of relationships among types of figures and critical components of figures. The explorations can be supported by varied instructional strategies, including the use of physical or drawn models of geometric figures and the use of geometric software. Students can use geoboards, dot paper, multiple-length cardboard strips with hinges, or drawing software to create two-dimensional figures and to make conjectures about the features and components of the figures or about relationships among different types of figures. For example, purposeful explorations with dynamic geometry software can help students discover subtle relationships among types of quadrilaterals. Representative results of such an investigation are summarized in figure 6.2 and provide a foundation for a conjecture that the diagonals of a square are perpendicular and of equal length, the diagonals of nonsquare rectangles are of equal length but not perpendicular, the diagonals of nonsquare rhombi are perpendicular but not of equal length, and the diagonals of other parallelograms are of unequal length and not perpendicular.

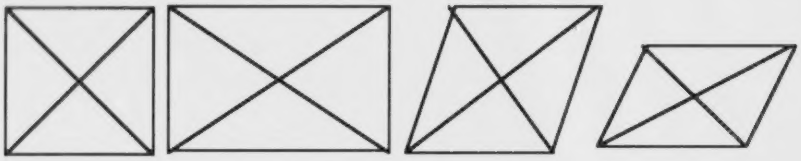


Figure 6.2.Diagonals of quadrilaterals.

Students also need experience developing clear, concise, and compelling explanations and justifications for conjectures they form on the basis of geometric investigations in the middle grades. For example, students could justify a claim that two triangles are similar by identifying an invariant ratio of similarity that holds for all corresponding sides. Or they use their knowledge regarding the interior angles of different regular polygons to explain why only triangles, squares, and hexagons can be used to make a regular tiling of the plane. Experience in communicating such reasoning accurately and clearly is preparation for the creation or understanding of more formal proofs in subsequent schooling.

Many investigations in middle grades geometry can be motivated by or connected to other school subjects. Nature, art, and the sciences provide opportunities for observation and subsequent exploration of geometry concepts and patterns as well as for students to gain an appreciation and understanding of the beauty and utility of geometry. For example, the study of golden rectangles (i.e., rectangles in which the ratio of the lengths is a golden ratio) can connect to students' reading of ancient fables in language arts class or through a consideration of ancient Greek architecture. Students could try to find other appearances of the golden ratio in art or nature (e.g., the type of growth exhibited by the shell of a nautilus). In this case, students would not only connect mathematics to other topics, but they could also develop their proportional reasoning and measurement skills as they work with the golden ratio. Tessellation of the plane and projection in drawing are other topics that provide opportunities for geometry investigations connected to art and nature. Through their work with tessellation students can examine the characteristics of various shapes, such as the sizes of the interior angles of regular polygons. Projective geometry is another topic that connects geometry to art and nature. In drawing projections students work with ratios and similar figures.

- **Coordinate Geometry and Other Geometric Models**

Geometric and algebraic representations of problems can be linked using coordinate geometry. Students can increase their mathematical understanding of the properties of two-dimensional geometric figures by representing them using coordinates in a plane and interpreting them algebraically, using distance and slope. For example, coordinate geometry lends itself well to developing a more analytic understanding of parallel and perpendicular lines. Students can use the coordinate plane to construct and explore properties of parallel and perpendicular lines using slope and distance. To extend this knowledge to figures and their relationships, the teacher could ask students to graph given coordinates for vertices of parallelograms and then have students conjecture about and verify slope relationships for pairs of perpendicular, nonperpendicular intersecting, and parallel lines.

The Pythagorean Theorem is one of the most useful relationships in school mathematics, and students should be able to use it flexibly by the end of eighth grade. In the middle grades, the Pythagorean Theorem introduces students to irrational numbers, helps determine the relationships between the legs of 30-60-90 degree triangles and isosceles right triangles, and can be used to indirectly determine distances that would be difficult to measure directly either on the coordinate plane or in the context of other problems. The history of the Pythagorean theorem can also be interesting for students and provides opportunities for connections between mathematics and social studies. Furthermore, the accessible nature of a number of proofs that have been suggested for the Pythagorean theorem provides an excellent opportunity for experience with mathematical proof. Students could try to replicate some of the visual proofs of the theorem using geometry software or paper-cutting procedures and could discuss the reasoning behind these. As is discussed in the Problem Solving section of this chapter, the relation lends itself to further explorations and allows for interesting generalizations and conjectures that are within the reach of middle grades students.

In addition to their work with coordinate geometry, middle school students will benefit from experience with other representational models, such as networks. These can provide students with additional tools to use in analyzing and solving problems.

• Transformations

Transformational geometry offers students another lens through which to investigate and interpret geometric figures. Students can use MIRAs, folded paper, or mirrors to examine line symmetry, to investigate the images of figures under various transformations, and to discern relationships among combinations of flips, turns, and slides. Computer software or graph paper can also be used for these explorations and for examining rotational symmetry and magnification.

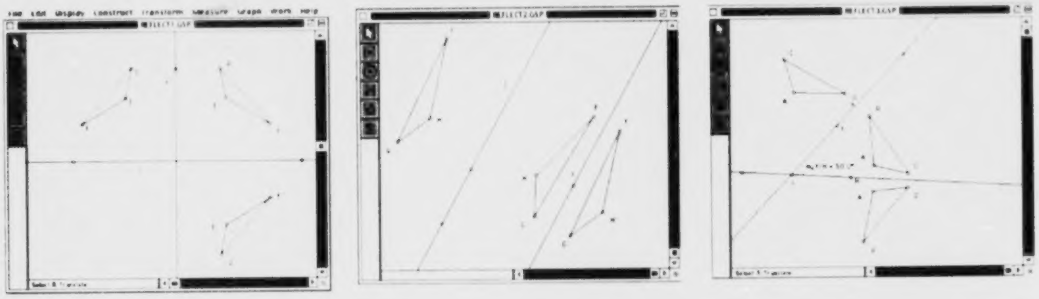


Figure 6.3. Composing flips.

Students could use dynamic geometry software to investigate the general result of successive flips in two different lines. Figure 6.3 shows three possible situations that could be examined by students: when the two lines are perpendicular, when they are parallel, and when they intersect in general position. Working in pairs or small groups, students can flip a triangle over one line, and then flip the image over another parallel or intersecting line (perpendicular and non-perpendicular). They could be encouraged by their teacher to conjecture about what single transformation, if any, would have exactly the same effect on the original triangle and then to verify their conjecture. In particular, students can develop and justify many interesting conjectures such as: doing consecutive flips through two parallel lines is equivalent to a slide twice the distance between the lines or doing consecutive flips through two intersecting lines is equivalent to a turn centered at the point of intersection and through an angle twice the angle formed by the lines. The use of dynamic geometry software in this case facilitates the generation of different examples within each case, thereby enabling students to examine cases and test conjectures with ease, but graph paper could also be used.

Using transformations students can explore and understand similarity and congruence relationships between figures. That is, students can verify congruence relationships between figures and their images after sliding, turning, or flipping, and similarity relationships between figures and their images after magnification. In this way, transformations can provide a base for defining and understanding congruence and similarity.

Students can connect their geometric and algebraic understanding of transformations by using coordinates in the plane. In using coordinates to locate figures and their images under various transformations, students can apply their knowledge of parallel and perpendicular lines, of distance and slope, and of ratio and proportion. An array of problems about geometric figures based on their properties can be posed and solved in this way (e.g., identifying a transformation that would make one figure the image of another, finding lines of symmetry for figures, or locating the image of a figure under a specified transformation).

- **Visualization**

Student visualization skills and spatial reasoning are fundamental to understanding relationships in geometry. It is important for middle grades students to develop an understanding of the relationship between three-dimensional shapes and their two-dimensional representations, so that students will be able to both interpret two-dimensional representations and to draw them. Instructional activities that ask students to interpret or draw different views of buildings, such as the base floor plan and front and back views, using dot paper, can be useful in developing this understanding. They should have the opportunity to build three-dimensional objects from two-dimensional representations, draw or sketch figures given a geometric description; or write a description, including geometric properties, for a given figure. Experience with models of three-dimensional figures and their two-dimensional nets is also useful in this regard, and it may also help develop students' understanding of surface area. Students also need experience with complex two- and three-dimensional figures that they examine, build, compose, and decompose. Such experience can be obtained using a variety of media, including paper-and-pencil sketches, geometric models, and geometry software.

Standard 4: Measurement

Mathematics instructional programs should include attention to measurement so that all students—

- **understand attributes, units, and systems of measurement;**
- **apply a variety of techniques, tools, and formulas for determining measurements.**

Elaboration: Grades 6–8

Students bring years of diverse experiences with measurement to middle school, not only from prior classroom instruction but also from using measurement in their everyday lives. Measurement concepts and skills are used throughout the middle grades in mathematics, other content areas, and school activities. When painting backdrops for a school play, members of the drama club may measure length (to divide panels into sections of a reasonable size) as well as compute area and use proportional reasoning (to decide how much paint or colored paper to buy). In planning refreshments for a school dance, student council members may need to estimate how many gallons of lemonade to buy. Athletes on the track team may be curious about how their times for sprints or cross-country runs compare with Olympic times for similar events. Measurement instruction is best provided in context, because contexts enhance understanding and provide natural connections from one mathematics topic to another as well as from mathematics to other subject areas. Furthermore, connecting the physical act of measuring with visual and symbolic representations of the measurements leads to deeper understanding of both the concepts and processes of measurement.

Measurement should be emphasized throughout the school year, rather than treated exclusively as a separate unit of study. Important aspects of measurement in the middle grades include choosing and using compatible units of measure for the attributes being measured, selecting appropriate units and scales on the basis of the precision desired, and solving problems using the perimeter and area of two-dimensional figures and the surface area and volume of three-dimensional figures. Students should also become proficient at measuring angles with protractors and using ratio and proportion to solve problems involving derived measures and unit rates.

Focus Areas for Grades 6–8

- **Understand attributes, units, and systems of measurement**

  In grades 6–8, all students should—

  - select appropriate units and scale to estimate and measure angles, perimeter, area, surface area, and volume;
  - understand both metric and customary systems of measurement, including relationships among units of the same system.

- **Apply a variety of techniques, tools, and formulas for determining measurements**

  In grades 6–8, all students should—

  - be proficient in measuring angles in plane figures;
  - develop and use formulas for the perimeter and area of parallelograms, trapezoids, circles, and simple composite figures;
  - develop and use formulas for the surface area and volume of prisms, pyramids, and cylinders;
  - select techniques and tools to measure accurately with levels of precision appropriate to the situation;
  - use ratios and proportions to solve problems involving scale factors;
  - determine an appropriate scale and use scale drawings or models in applications;
  - solve simple problems involving rates and derived measures (e.g., miles per hour).

Discussion

- **Measurement Attributes, Units, and Systems**

In the middle grades students build on their elementary experiences with measurement attributes such as angle, length, area, and volume; units of measure; and systems of measurement. Coming into the middle grades, students should be comfortable working with English units (e.g., miles or quarts) and metric units (e.g., km or liters). And they should be familiar with common benchmarks within these systems, such as 1,000 mm = 100 cm = 10 dm = 1 m and 36 inches = 3 feet = 1 yard. In the middle grades they should become proficient in representing measures in different units within a system and estimating measures (e.g., 100 yards is the length of a football field, a meter is approximately the height of a doorknob, 30°C might be the temperature on a hot summer day). When asked to estimate the height of the classroom door or the capacity of a pitcher, students should be able to suggest reasonable measurements. Even though there is no need for students to memorize metric and customary measurement conversion equivalencies, they should know useful benchmarks between systems, such as a quart is a little less than a liter, a yard is a little less than a meter, and so on.

- **Measurement Techniques, Tools, and Formulas**

Students in grades 6–8 should become proficient in measuring angles and in understanding a variety of angle relationships. Angles are an essential component of many topics in the mathematics curriculum. Students use the measures of angles to describe and classify simple shapes such as quadrilaterals and triangles, as well as to determine relationships in more complex polygons and three-dimensional figures. A firm understanding of angle measurement is required to draw and interpret circle graphs. Middle grades students develop a repertoire of benchmark angles including right angles, straight angles and other angles useful in work with triangles (e.g., 30°, 45°, 60°). They should be able to offer reasonable estimates of the measure of any angle between 0° and 180°. Students should become familiar with the characteristics of angles formed by intersecting lines, particularly when sets of parallel or perpendicular lines are involved, and with using relationships among the angles formed to solve problems.

Students can consider angle relationships (e.g., vertical, alternate interior, alternate exterior, corresponding, supplementary, complementary) in two versions of parallelogram ABCD (figure 6.4). Using dynamic computer software or graph paper and protractors, students can measure angles and conjecture about angle relationships. Students can investigate patterns in angle measures through questions such as the following: In the two parallelograms ABCD, which pairs of angles remain supplementary? What angle relationships remain the same? What general statements can you make about angles in parallelograms?

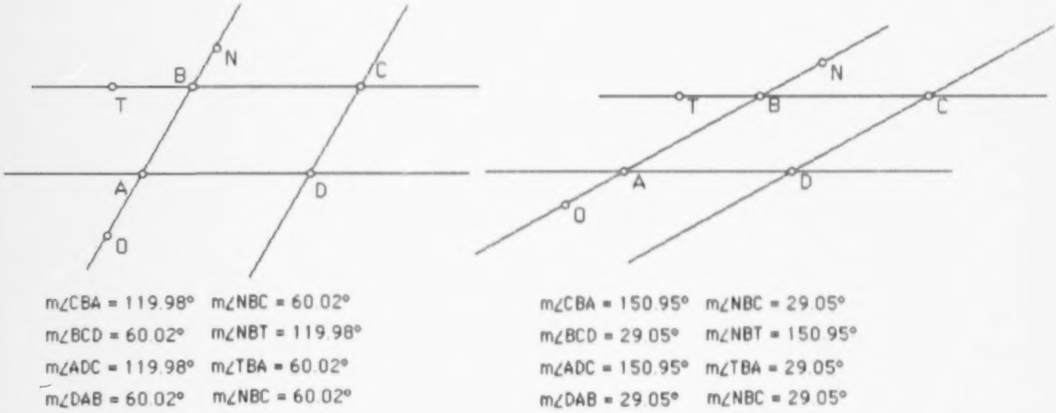


Figure 6.4. Two possibilities for parallelogram ABCD.

Another goal in grades 6–8 is for students to become proficient at determining the perimeter, area, and volume of many figures and to use this proficiency in problem situations. Middle grades students formalize techniques acquired in the early grades for finding areas of rectangles and triangles. They develop and use area formulas for parallelograms, trapezoids, other polygons, circles, and related composite figures. In the example in figure 6.5, students have to apply what they know about finding the area of triangles and rectangles to find the area of several trapezoids.

> Students in a seventh-grade mathematics class are working on a project to design and construct a scale model of a house within a fixed price range. They must determine the costs of materials and the final cost of the house to make sure that it falls within the price range. The three students working on the roof need to find the total area of the roof before they can price the shingles. This is where they encounter a problem. The roof will be made up of two pairs of congruent isosceles trapezoids and a rectangle, and they have no idea how to find the area of a trapezoid.
>
> The students suggest different methods to find the area using their prior knowledge about the areas of related figures (see figure 6.5). One student suggests using grid paper to cover and count; another wants to decompose the trapezoids into triangles and a rectangle and use

known formulas to find the area. The third student decides to transform the isosceles trapezoids into a rectangle by removing a triangle from one side and placing it on the other. In each case, the students find approximately the same area for the trapezoids and use this information to develop an estimate for the cost of the shingles.

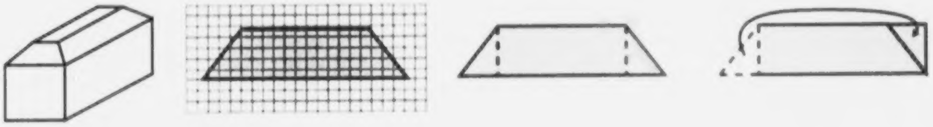


Figure 6.5. Finding the area of a trapezoid.

Student discussion about the three methods used in this problem can help them to derive a general formula for the area of a trapezoid. In the future, some students may recall the formula when they need to find the area of a trapezoid, and others may quickly replicate one of the methods used in this project.

Similarly, from knowing how to find the area of a circle and the area of a rectangle, students can develop a formula for the surface area of a cylinder. By constructing models and looking for patterns and relationships, middle grades students can also develop and use formulas for the surface area of prisms, pyramids, cylinders, and special cones, and for the volume of right prisms and cylinders.

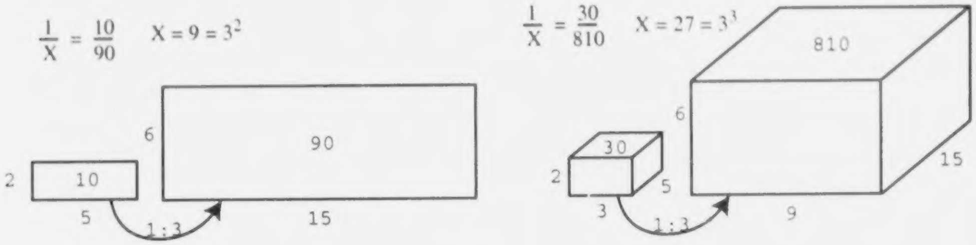


Figure 6.6. Scale factors for area and volume in similar figures.

Another important focus in the middle grades is having students investigate and generalize relationships among length, area, and volume in similar two- and three-dimensional figures. Ratio and proportion are important tools in this process. Students need ample experiences drawing similar figures or scale models, measuring attributes, and using proportions to compare measures. For example, in figure 6.6 students construct a 2 x 5 rectangle and then use a ratio of 1:3 to construct a similar rectangle with dimensions 6 x 15. Students find that the ratio between the areas of the rectangles is 1:9 or $1{:}3^2$. They can also create similar prisms and find that the ratio between volumes is 1:27 or $1{:}3^3$. Students can investigate other similar rectangles and prisms with different scale factors to determine if these relationships appear in other cases. They can reason about why these relationships are reasonable, and eventually generalize them: figures whose linear measurements have a ratio 1:a will have areas with ratio 1:a^2 and volumes with ratio 1:a^3.

The "Circles in a Square" problem (figure 6.7) is another interesting task that can engage students in thinking about similarity and proportional relationships. In order to see the equivalence of the areas of the dark portions of the square in each of the three representations shown as A, B, and C in figure 6.7, students could compute the areas to obtain $(64 - 16\pi)$ square units in each drawing. The computation for drawings B and C can be done in more than one way, and students could check to be certain that the same answer is obtained each way. But it is also possible to solve the problem without numerical calculation if students recognize the circles as similar figures and are familiar with the generalization discussed in the previous example. Because the square in drawing B is divided into four circles, each similar to the circle in drawing A, but with 1/2 the radius, the total area of the four circles must be the same as the area of the circle A. Similar reasoning applies in drawing C to the sum of the areas of the 16 circles (each with a radius 1/4 the circle in A). This solution implicitly uses the formula for the area of a circle by recognizing that the radius of the circle is squared to obtain its area. Yet another solution approach uses proportional thinking; imagine each of the circles in drawings B and C as being inscribed within its own smaller square as in drawing A. Since the scale factor for the side of the square is 1/2 in drawing B and 1/4 in drawing C, then the scale factor for the areas will be 1/4 and 1/16, respectively, and the equivalence of the total areas is obtained.

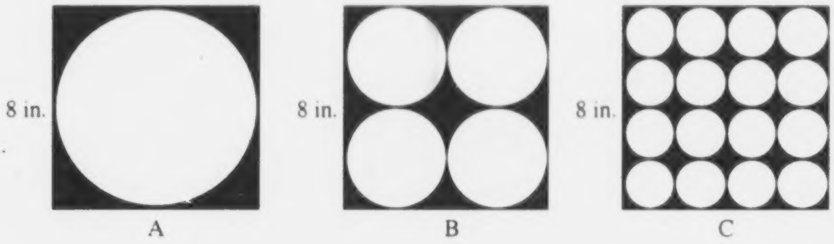


Figure 6.7. Circles in a square.

Students become more proficient in selecting appropriate units of measure on the basis of the precision of the instruments, the relative and absolute error of the instruments, and the context of the situation. For instance, if students are measuring the length of their classroom in meters, they should understand that an error of a few meters is not acceptable because a few meters is a large percentage of the total length of the room. However if the length were measured in centimeters, an error of a few centimeters would be acceptable. Similarly, when students find that the length of a book, to the nearest quarter inch, is 12 1/4 inches, they should realize that the measurement could be off by 1/8 of an inch. The absolute error in the measurement is ±1/8 inch. They should understand that a measurement will be precise to one-half the unit used in the measurement.

Finally, students in the middle grades investigate numerous situations that involve derived measures (typically, rates found by dividing one measure by another). For example, if they know the distance (in kilometers) traveled in a given time (in hours), students can calculate speed in kilometers per hour. Or they can calculate their heart rate (beats per minute) or the unit price of snacks that are packaged by the dozen (dollars per single pack). The notion of unit rate becomes a very helpful tool for calculating and comparing derived measures.

Standard 5: Data Analysis, Statistics, and Probability

Mathematics instructional programs should include attention to data analysis, statistics, and probability so that all students—

- **pose questions and collect, organize, and represent data to answer those questions;**
- **interpret data using methods of exploratory data analysis;**
- **develop and evaluate inferences, predictions, and arguments that are based on data;**
- **understand and apply basic notions of chance and probability.**

Elaboration: Grades 6–8

The study of data analysis and statistics in the middle grades provides opportunities for students both to use what they have learned elsewhere in their study of mathematics and also to learn new ideas. Similarly, the study of probability both builds on students' knowledge of ratio and proportionality and develops new understandings of concepts and procedures associated with chance.

In the middle grades, students should engage in and employ the full process of data investigation: posing questions, collecting data, organizing data, analyzing data, interpreting data, and answering questions. Data analysis in the middle grades provides a rich context for students to pose and solve problems; to engage in purposeful measurement, calculation and estimation; and to gain experience with a variety of graphical representations. Typically the numbers encountered in data applications are contextualized, and so these application settings enable students to make sense of numerical quantities and relations among quantities. Such experience helps students become thoughtful users of mathematics.

Students' work with data analysis and statistics in grades 6–8 draws on and integrates their knowledge of ratios, fractions, decimals, percents, graphs, and measurement. At the same time, students develop or extend their understandings of ideas that are central to the study of statistics, such as data distribution, central tendency, and variance. In the middle grades, students extend their work with data in earlier grades by analyzing more complex data sets, answering more difficult questions, and investigating associations between variables. New representational forms are added to students' repertoire (e.g., box-and-whisker plots) in order to allow more complex considerations of data distributions. The analysis of scatter plots for related variables allows students to develop linear approximations (lines of best fit) for the plots, as part of their consideration of linearity in the middle grades. Students' work with data can be connected to their study of probability, both through the determination of experimental probabilities and through the use of probabilistic inference from random samples.

Focus Areas for Grades 6–8

- **Pose questions and collect, organize, and represent data to answer those questions**

  In grades 6–8, all students should—

  - design experiments and surveys, and consider potential sources of bias in design and data collection;
  - recognize types of data (e.g., categorical, count, continuous or measurement, and organize collections of data;
  - choose, create and utilize various graphical representations of data (line plots, bar graphs, stem-and-leaf plots, histograms, scatter plots, circle graphs, and box-and-whisker plots) appropriately and effectively.

- **Interpret data using methods of exploratory data analysis**

  In grades 6–8, all students should—

  - find, describe, and interpret mean, median, and mode as measures of the center of a data set; know which measure is best to use in particular situations; and understand how each does and does not represent the data;
  - describe and interpret the spread of a set of data using tools such as range, interquartile range, and box-and-whiskers graphs;
  - interpret graphical representations of data, including description and discussion of the meaning of the shape and features of the graph, such as symmetry, skewness, and outliers;
  - analyze associations between variables by comparing the centers, spreads, and graphical representations of related data sets;
  - examine and interpret relationships between two variables using tools such as scatter plots and approximate lines of best fit.

- **Develop and evaluate inferences, predictions, and arguments that are based on data**

  In grades 6–8, all students should—

  - develop conclusions about a characteristic in the population from a well-constructed sample;
  - through simulations, develop an understanding about when differences in data may indicate an actual difference in the populations from which the data were collected and when the differences may result from natural variation in samples;
  - use data to answer the questions that were posed, understand the limitations of those answers, and pose new questions that arise from the data.

- **Understand and apply basic notions of chance and probability**

  In grades 6–8, all students should—

  - make judgements about the likelihood of uncertain events and be able to connect those judgements to percents or proportions;
  - understand what it means for events to be *equally likely* and for a game or process to be *fair*;
  - compute simple probabilities using appropriate methods, such as lists, tree diagrams, or area models;
  - identify complementary, mutually exclusive, independent, and dependent events and understand how these relationships affect the determination of probabilities.

Discussion

- **Produce, Organize, and Represent Data**

Middle grades students often find it interesting to pose questions and collect data about their peer group. For example, students in these grades often collect data about their personal interests (e.g., their favorite foods, sports, or music). Such data collection can provide an occasion for considering the appropriate design and use of surveys.

Opportunities to work with data and to think about the design of careful data collection are also often associated with cross-disciplinary activities, such as science experiments. For example, in science class students might study the relationship between light absorption and heat by designing an experiment in which they indirectly measure the heat generated when light is absorbed by collecting and examining data on the melting time of ice when it is placed under a lamp on mats of various colors (e.g., white, gray, black). Students might ask questions like these: How can we make sure we always use the same amount of ice? What's the most accurate way to measure the melting time? How might room temperature affect melting time? Does it matter if there is other light in the room besides the lamp? How many times should we run the experiment in order to get an accurate measure? From such experiences, students see both the difficulties and possibilities inherent in designing useful data collection.

Once data have been collected, they need to be represented or displayed in some way. This gives students opportunities to consider the relative strengths and weaknesses of various forms of data display. Suppose, for example, they were trying to determine which of two brands of frozen "supreme" pizza has the most pepperoni. The students decide to collect data on the number of pepperoni slices on a sample of each brand of pizza by visiting several supermarkets and examining samples in the frozen food section. Here is a compilation of the data collected by students in the class:

Brand 1: 4,2,3,3,1,5,2,5,5,7,10,2,8,13,1,5,5,12,7,9,4,13,2,4,3

Brand 2: 3,8,10,6,4,11,9,8,5,9,7,10,3,10,7,5,9,8,7,5,6,4,7,8,8

After the data are organized and displayed in this way, the students discuss their findings and realize that it will be difficult to answer the main question they are trying to answer unless they find a better representation for the data. Their teacher suggests that they try several different graphical representations of these data in order to see which might be helpful. She asks one group to display the pizza data using a bar graph, another group to use a histogram, and a third group to use a new kind of representation called a box-and-whisker plot. The results of their work are displayed in figure 6.8. In the ensuing discussion of these forms of representation, the students see that their bar graph depicts the distribution and variation within the sample for each brand, their histogram provides the average for each brand, and their box-and-whisker plot gives both the average and the variation for the distribution of pepperoni slices for the two brands of pizza. Through thoughtful reflection on experiences with data collection, organization, and display, students can understand that different representations require different conceptualizations of a data set and yield different kinds of information. Thus, different representations are likely to be useful for different purposes.

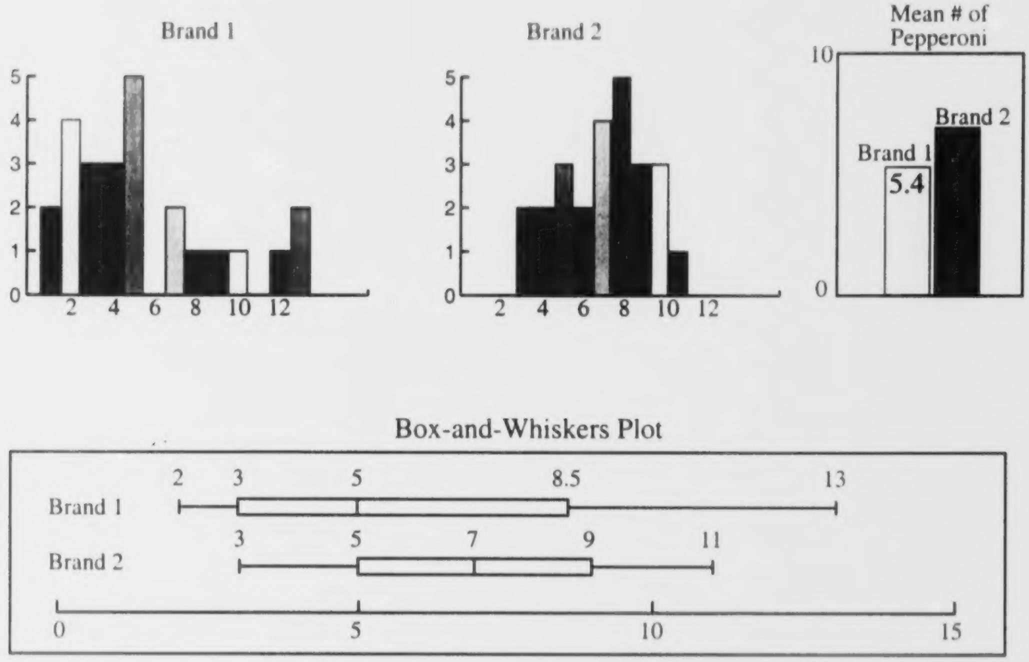


Figure 6.8. Comparing two brands of pepperoni pizza.

Data from newspapers, magazines, brochures, and the World Wide Web are readily available and can easily be brought into the classroom for investigation and discussion. When reviewing others' data, discussion should consider why the author chose particular representations and scales and whether these choices enhance the point the author was trying to make. Students should understand that the representation of data is a means of communication and must be chosen with some thought. If possible, students should also consider how the data were collected, how accurate were the measurements used in collecting the data, and could the data be biased.

- **Describe, Analyze, and Interpret Data**

An important part of any statistical process is to describe the shape, spread, and measures of center for a data set. At the beginning of middle grades, students expand their understanding of measures of center to include what the mean represents and what it tells us about a set of data. They come to understand mean as a number that "evens out" or "balances" a distribution. They have experiences that ask them to consider the properties of mean and median, particularly the influence of extreme values on each measure. Discussion should focus on which measure of center is the "best" representation for a particular data set and why.

In the case of the pizza data, if students have data from more than two varieties of pizza, some students might claim the best pizza is the one with the largest mean number of pepperoni. Others might argue that the pizza with the highest median is best because you will get that amount or more at least half the time. And still others might argue for the pizza with the highest mode because that is the most likely number of pepperoni on a pizza. Students should understand that not all of these responses are appropriate. The mode, in this instance, does not take into account the entire data set and thus is a poor measure of center. The mean and median are better choices because they consider all of the data within the set.

Middle grades students need to develop the habit of looking at data in depth. With additional experiences, they increase their ability to describe the shape of the data and make connections between the shape and the data set. Students talk about data being symmetrical or skewed. They identify the range, but they also look at inter-quartile ranges and compute to see if any data points are outliers. With more complex problem situations, looking for patterns in the data distribution should also lead students to consider associations between variables.

With problems involving paired measurements, such as the height from which a ball is dropped and the height it reaches on its first bounce, students can study associations between variables. Such data can be represented using scatter plots. Students can informally find lines of best fit, setting the stage for more formal study in high school. In such cases, they should consider what the best-fit line indicates about the data. In discovering a correlation between paired measurements, it is important that students realize that a correlation does not necessarily imply causality.

- **Inference and Prediction**

Making inferences and predictions requires students to use graphic representations and statistics to answer questions. Middle grades students come to understand how samples relate to populations. Students gain information from this relationship to make sense of the data and make decisions or see possible consequences. They discuss and experiment with different ways to select samples. From these experiences, they come to realize the importance of random samples. Students also should come to recognize that the accuracy of a sample statistic improves with the size of the sample.

As students address issues related to sampling and design, they come to understand that well-constructed samples enable them to develop inferences or draw conclusions about the population with more certainty. When asking questions like, "How many video games are played in one week?" students should consider the population, the sample size, and any bias that may occur before collecting the data. When using the data to answer questions, new questions may arise. These new questions come from realizing the limitations of the responses. For example, follow-up questions like the following may be asked: How long do students play a particular video game during a week?, What types of video games are played?

Data are often used to compare populations. By understanding that there is natural variation in data, students understand that if the centers of two sets of data are relatively close, there may be no difference in the populations on the variable measured. Similarly, they understand that if there is a moderately large difference in the centers, the populations may be quite different. For example, if the mean number of pepperoni on two different brands of pizza is close, this is not enough to conclude one brand is better than the other. By comparing the data, students needed to decide if the difference between the means is sufficiently large to draw such a conclusion. This prepares students for the study of statistical hypothesis testing in high school. Simulations can help students gain intuition about how much variability to expect between means of samples drawn from the same population and how sample size affects this variability.

- **Probability**

In grades 6–8 students' experiences with data and their emerging understandings of ratio and proportionality should be intertwined with their study of probability. When students analyze data from a random sample, they can use their observations to make probabilistic predictions about the population. In computing theoretical probabilities, they use their understandings of ratios and proportions to predict the likely configuration of occurrences in the population. In their study of theoretical and experimental probability in the middle grades, students learn to use the idea of relative frequency to determine simple probabilities. In addition to computing probabilities, students need considerable time and experience to build a solid conceptual foundation for their work in probability. For example, they need to come to understand notions of *equally likely*, *fairness*, *randomness*, and *relationships* among events, such as independent and dependent events, complementary events, or those that are mutually exclusive. These experiences prepare students for the further study of statistics and probability in high school (Burrill and Romberg 1988).

Although the computation of probabilities can seem to be a fairly simple exercise in working with fractions, there are many deep conceptual issues with which students need to grapple in order to understand probability. In fact, it is an area of mathematics in which many adults as well as students harbor misconceptions (Konold 1998). For example, many find it difficult to understand the difference between outcomes being determined versus outcomes being likely (Konold 1989). That is, some students perceive each individual trial of an experiment not as one of many such possible results but rather as *the single outcome* of the experiment. Students who think this way will have a very difficult time making sense of theoretical probability calculations based on a sample space containing equally likely outcomes. This is one of many such conceptual obstacles that teachers and students are likely to encounter in their work with probability (Bright and Hoeffner 1993). For this reason, it is important for students to have extensive experience with activities in which they deal with uncertainty and to have frequent opportunities to compare their probabilistic thinking with that of others (Shaughnessy and Bergman 1991). Computer simulations (or those enacted physically) can also help students develop intuitions and confront misconceptions.

Standard 6: Problem Solving

Mathematics instructional programs should focus on solving problems as part of understanding mathematics so that all students—

0. build new mathematical knowledge through their work with problems;

1. develop a disposition to formulate, represent, abstract, and generalize in situations within and outside mathematics;

2. apply a wide variety of strategies to solve problems and adapt the strategies to new situations;

3. monitor and reflect on their mathematical thinking in solving problems.

Elaboration: Grades 6–8

Problem solving is the process by which students experience the power and usefulness of mathematics in the world around them. It is also a method of inquiry and application, interwoven throughout mathematics to provide a consistent context for learning and applying mathematics. Our goal is to help all students become flexible, autonomous, and effective problem solvers. There are four key aspects to the multifaceted process of problem solving. The first is a positive disposition or attitude toward problem solving. Effective problem solvers are curious about problem situations and interested in exploring them, representing them mathematically, making conjectures, and investigating the reasonableness of those ideas. The second key aspect of the problem solving process is knowledge. Content knowledge, and the ability to use that knowledge in a variety of situations, is required for students to solve challenging problems. The act of solving problems can also help students investigate new ideas and extend their understanding of previously learned concepts. The third key aspect of the problem-solving process is strategies. Seasoned problem solvers have ready access to a diverse collection of general but powerful strategies to employ when confronted with unfamiliar problem situations. Students benefit from purposeful instruction and diverse experiences related to strategies. The final key aspect of the problem-solving process is monitoring. It is important for students to be aware of their own strengths and weaknesses, think about their own thinking as they solve problems, plan frequently and efficiently, and engage in ongoing self-assessment.

What does problem solving look like in grades 6–8?

Problem solving in grades 6–8 should be a vehicle for mathematical learning. Students can learn about mathematical concepts by working through problem situations that embody important aspects of those concepts (Schroeder and Lester 1989). Consider this problem: Find all possible shapes with an area of 14 square units and a perimeter of 24 units, if each shape must consist of a collection of unit squares where at least one side of every square must share a complete side with another square. Suppose the students given this challenge have been previously introduced to the notions of area and perimeter for rectangular shapes. A reasonable approach to the problem is to begin by experimenting with a few shapes, using graph paper or 14 square cutouts that they can manipulate. It soon becomes apparent that haphazard experimentation is not likely to produce a complete solution. Students should realize that they need to develop a systematic way to keep track of the shapes that have been tried so that they obtain all possible solutions. They need to decide what makes a shape "different" so that they avoid duplication. Are the shapes different if one is a flip of the other? How do students identify strategies for creating new shapes from old that preserve certain qualities, such as area and perimeter? Skills such as these are important in a wide variety of problem situations. However, working on problems also deepens students' understanding of area and perimeter, helping them to recognize that there is no simple correlation between these two measures. Knowing the area of a figure does not determine the perimeter. Indirectly, this problem also involves students' knowledge of congruence and transformations that preserve distance.

By the middle grades, the universe of problem contexts should diversify. Increasingly, problems can grow out of situations in the world at large or from the investigation of mathematical ideas. Mathematics instruction in grades 6–8 should take advantage of the expanding mathematical capabilities of middle school students to include more complex problem situations that integrate topics such as probability, statistics, geometry, and rational numbers. Situations and approaches should build on and extend the mathematical understanding, skills, and language that students are acquiring. In addition, the problems should help students develop a variety of problem-solving strategies and approaches.

Problems that deal with proportionality, fractions, and linearity are particularly valuable in reinforcing students' understanding of these important middle school topics. Consider the following problem.

> A clear container is presented to students. The container has a number of red and green marbles. It has a top, which the students are not allowed to remove. The task is to approximate the number of each color marble and to justify their approximation. The students have access to an empty duplicate container, a number of marbles, and other tools, such as a ruler and scales. After spending some time thinking about the problem, some students may realize that if they weigh the container with marbles, weigh the empty container, and weigh the separate marbles, they can use that information to get a good estimate for the total number of marbles in the container. They can then count the number of red and green marbles that are visible and use that as an estimate for the proportion of red and green marbles in the container, allowing them to make an estimate for the total of each. Other students may go further, by shaking up the container and again counting the visible of red and green marbles. By repeating this process a number of times they can generate data that allows them to make a better prediction for the proportion of red and green marbles.

The previous problem has several aspects that should be found in many middle school problems. It requires that the students apply previously learned skills to approximate the total number of marbles. It combines several mathematical topics at different points in the problem-solving activity. And it requires the use of proportions and fractions, key learning objectives at these grades.

Middle grades students can deal with messier and more complex problems because they can use technology to alleviate much of the drudgery that—until very recently—often constrained school mathematics to considering problems with "nice numbers." Computers, calculators, and electronic data-gathering devices, such as computer-based laboratories (CBLs), provide methods for gathering and analyzing data that in years past might have been considered too much trouble to deal with. Similarly, classroom internet connections make it possible for students to look up facts and figures for use in posing and solving a wide variety of problems. For example, students might be interested in investigating whether it is cost-effective to recycle aluminum cans at their school, or they might explore data related to weather patterns in different regions. Graphing calculators and easy-to-use computer software enable students to move effortlessly between different representations of problem data and to compute with large quantities of data and with messy numbers, both large and small, with relative ease. As a result, problems in the middle grades can and should be very responsive to students' questions and interests.

How do students become problem solvers at grades 6–8?

Middle grades students should have ample opportunities for both independent and collaborative problem-solving experiences. Much can be learned through the exchange of ideas. Similarly, students in the middle grades can engage profitably in extended investigations, sometimes working for days on a single problem, and its extensions.

Students at this age should continue to develop their skills at explaining themselves clearly to others. To develop their problem-solving skills, they need to learn to reflect on their own work and to monitor their solutions. They should be expected to explain their ideas and solutions not only in words, but also using conventional mathematical symbols or other representations when appropriate. However, it is important for teachers to realize that none of these capabilities comes automatically just because students have reached middle school age.

There are four aspects of the problem solving process that are particularly useful for middle grades students: understanding the problem, devising a plan, carrying out the plan, and looking back (Pólya 1957). The teacher, through observing and questioning, can help students become aware of when and how they engage in each of these aspects while they are solving problems. Talking about these aspects can be helpful in getting students to better monitor their own actions during problem solving. Note that these aspects are not necessarily linear. Expert problem solvers move back and forth among all four of these aspects repeatedly as they work through a problem (Lambdin Kroll, and Miller 1993).

Although it is not the main focus of problem solving in the middle grades, when students learn *about* problem solving they become familiarized with a number of problem-solving heuristics, such as looking for patterns, solving a simpler problem, making a table, and working backwards. These are general strategies that are useful when no known approach to a problem is readily at hand. Many of these heuristics will already have been widely used in the elementary grades, but middle grades students need additional experience and instruction in which they consider how to use these strategies appropriately and effectively.

Students also should be encouraged to self-monitor and self-assess as they proceed through the problem-solving process. Good problem solvers recognize what they know and what they don't know, what they are good at and not so good at, and as a result they can use their time and energy wisely when solving problems. They plan more carefully and more effectively and take time to check their progress periodically while solving problems. These habits of mind are important not only in making students better problem solvers, but also because they help students become better learners of mathematics.

How can problem solving promote mathematical understanding in grades 6–8?

There are several reasons students should reflect on their own problem solving, and consider how it might be modified, elaborated, streamlined, or clarified. Through guided reflection, students can focus on the mathematics that was involved in solving a problem, thus solidifying their understanding of the concepts involved. They can learn how to generalize and extend problems, leading to an understanding of some of the structure underlying mathematics. Students should understand that the problem-solving process is never finished until they have looked back at their solution.

For example, consider the Pythagorean theorem. Geometrically, it tells us that if we build squares on the legs and hypotenuse of any right triangle, then the areas of the squares on the legs will together sum to the area of the square on the hypotenuse. This is summarized with the formula $a^2 + b^2 = c^2$. If students have developed the habit of looking back at problems and asking "what if?" they might naturally ask questions like the following: What would happen if we built something other than squares on the sides of right triangles? What if we built equilateral triangles? Or isosceles right triangles? Or regular hexagons? Or semicircles? Will the areas still sum in the same way?

Exploring with dynamic software, students can discover that the areas do appear to sum the same way. Students who are familiar with finding areas of circles and triangles can actually verify these relationships for particular semicircles, or particular right triangles, such as the 3-4-5 right triangle, an isosceles right triangle, or the 30°-60°-90° triangle.

Students can be encouraged to make conjectures about further generalizations and to attempt to verify those conjectures. For example, they might conjecture that an extension of the Pythagorean theorem holds for any regular polygons (or, more generally, for any similar figures) constructed on the sides of a right triangle (Brown and Walter 1983). Students should be made aware that they have not actually proven the more general statements simply by observing the regularities and stating the conjectures. However, exploring conjectures and formulating informal logical arguments takes students a long way toward developing habits of mind that are mathematically powerful (Moses, Bjork, and Goldenberg 1990).

By reflecting on their solutions, such as in this Pythagorean extension, students use a variety of mathematical skills, develop a deeper insight into the structure of mathematics, and gain a disposition toward generalizing. The teacher plays an important role in ensuring that classroom discourse continues until a variety of solution paths have been considered and those paths have been discussed, understood, verified, and evaluated. It should become second nature for students to talk about connections among problems; to propose, critique and value alternative approaches; and to be forthright in explaining how they monitor, verify, and might extend their approaches to solving problems.

What is the teacher's role in developing problem solving at grades 6–8?

Much of the teacher's role in problem-solving instruction is embedded in the discussion above. Some specific suggestions for middle grades teachers include the following: Let students choose some of the problems they will try to solve; challenge students with problems that have more than one answer (or no answer), as well as problems with unique answers or with extraneous information or insufficient information; brainstorm with students to identify interesting and worthwhile problems and to model mathematical ways of thinking about problem posing; encourage communication and collaboration among students, and record and honor spontaneous what-ifs and conjectures. For example, naming a problem, conjecture, or solution method after its student inventor (e.g., referring to Kim's problem, Becca's conjecture, or Andrew's method) can be very motivating for students. Minimize time pressures when encouraging students to solve complex problems. Some problems may remain unsolved for days, thus providing the opportunity to model aspects of good problem solving such as perseverance, trying multiple approaches, and facing a challenge head-on.

Another important role for the teacher is fostering a positive disposition toward problem solving. A very important difference between successful and unsuccessful problem solvers is their attitudes and beliefs about problem solving, about themselves as problem solvers, and about ways to approach solving problems. Many students develop the faulty belief that all mathematics problems should be solved quickly and directly. If they look at a problem and do not immediately know what to do, these students often give up. They may develop an image of themselves as incompetent problem solvers, with long-lasting negative consequences.

It is important for teachers to help students learn what to do when confronted with unfamiliar problems; this is what problem solving is all about. Furthermore, many students believe there is just one "right" way to solve any mathematics problem. These students not only become dependent on the teacher or an answer key for verification of their success, but they also fail to appreciate the excitement and insight that can come from recognizing and connecting very different ways to solve a problem.

Standard 7: Reasoning and Proof

Mathematics instructional programs should focus on learning to reason and construct proofs as part of understanding mathematics so that all students—

- **recognize reasoning and proof as essential and powerful parts of mathematics;**
- **make and investigate mathematical conjectures;**
- **develop and evaluate mathematical arguments and proofs;**
- **select and use various types of reasoning and methods of proof as appropriate.**

Elaboration: Grades 6–8

Reasoning is an integral part of doing mathematics. It is connected to the other process standards. For example, students reason when they solve problems, and much of what they learn to communicate is connected to the process and product of their reasoning. Reasoning is also associated with all of the content standards. For example, students use mathematical reasoning when they form generalizations for patterns in the study of algebra, and when they use graphical and other forms of representation in order to solve problems involving data. Moreover, reasoning is intimately tied to teaching for thoughtful engagement and learning with understanding. When students' reasoning is made explicit, it both supports the construction of understanding and provides evidence of the current status of understanding. Thus, students in grades 6–8 should have numerous occasions and opportunities to engage in mathematical reasoning.

By the time they arrive in sixth grade, students have had experience creating and evaluating some mathematical arguments. In the middle grades they will sharpen their skills of mathematical argumentation, including more abstract and hypothetical components. They will learn to search for generalization and acquire a broader repertoire of representational tools to support their claims and justifications. In addition, they should develop a keener sense of audience in communicating their reasoning formally or informally, and they should develop more rigorous criteria for judging the adequacy of mathematical arguments. Middle grades students gain experience with many forms of reasoning, including recursive, inductive, deductive, and spatial reasoning. And proportional reasoning is of particular importance in the middle grades. Although mathematical argumentation in the middle grades lacks the formalism and axiomatic structure often associated with mathematical proof, it shares many of its features, including the formulation of a plausible conjecture, the testing of that conjecture using mathematical evidence, and the display of the associated reasoning for evaluation by others.

What does reasoning and proof look like in grades 6–8?

In the middle grades, students should frequently be asked to explain relationships involving mathematical concepts. Often the search for relationships is pursued through the study of patterns. When searching for a generalization, students can consider or generate a set of specific instances, organize them, and look for a pattern. The identification of regularities underlying patterns or sequences is typically based on inductive reasoning. On the basis of observations made about portions of a pattern, conjectures can be made and then tested using inductive or deductive inference.

Consider the work of a group of students in a sixth grade class who were investigating patterns in figurate numbers (Malloy 1997). They were first asked to generate the first five numbers in the set of triangular numbers, and they did so by using the visual structure of the numbers. They knew that triangular numbers could be represented by rows of dots in the form of equilateral triangles. In progressing from the first to the fifth, they added a row at the bottom of the triangle with one more dot than the bottom row in the previous triangle (see figure 6.9).

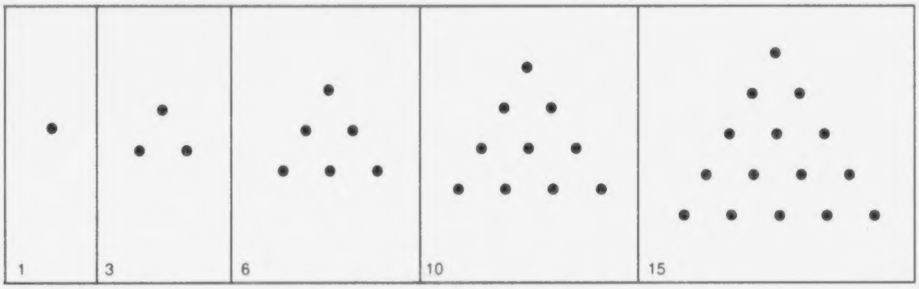


Figure 6.9. First five triangular numbers.

Next they were asked by their teacher to predict (without drawing) how many dots would be needed for the next triangular number. Reflecting on what they did to generate the sequence thus far, they quickly concluded that the sixth triangular number would have six more dots than the fifth triangular number. And they could see that this would seem to go on forever in the sequence of numbers. In fact, this led to a generalization about triangular numbers: Each triangular number is the sum of consecutive counting numbers. Thus, the students were engaging in recursive reasoning about the structure of this sequence of numbers, using the previous numbers to generate the next number. When the teacher next asked the students to find the 100th term in the sequence, they knew that it would be 100 more than the 99th term. But they did not know what that would be. Since they were not using a computer, it was not expedient to list the first 99 terms, so they needed another way to think about the pattern.

To facilitate their work, the teacher suggested that they make a chart to record their observations about the triangular numbers and the differences between numbers that they had detected thus far (see table 6.2). After studying the table for a few minutes, Tamika commented that there was a relationship between the differences and the number. She explained that the 2nd term was half of the product of 2 and 3, the 3rd was half of the product of 3 and 4, etc. For instance, the third term in the sequence is 6, the differences associated with 6 are 3 and 4, and 1/2 of the product (12) is 6.

Term	1st		2nd		3rd		4th		5th		6th
Number	1		3		6		10		15		21
Difference		2		3		4		5		6	

Table 6.2. Triangular numbers.

The teacher asked the group to check to see if this method would work to find the next number. Tamika said, "I think the 7th number is (7)(8)/2." Using their original sequence and the recursive pattern noted, the students tested her conjecture for the 7th term and found she was correct. Then the teacher asked for a conjecture regarding the 100th triangular number. Another student said, "If Tamika is right, the 100th number should be (100)(101)/2." The students discussed the value of Tamika's approach, which allowed the generation of a number in the sequence without having all the previous ones. In general, they agreed that this method was useful, although some were not convinced that it was correct, because it lacked the intuitive appeal of the recursive method they had used first. At this point the teacher decided to press the students to think about how they might verify Tamika's generalized rule for the sequence—half the product of the term number and the next term number, or $n(n + 1)/2$.

To help the students understand the validity of this generalization, she had them look at a simpler, related problem. She asked students to use "little Gauss's method" to find the sum of the first ten counting numbers. That is, she asked them to add matched pairs of numbers—1 + 10, 2 + 9, 3 + 8, 4 + 7, 5 + 6—from which they quickly saw that the sum of each pair equals 11. They then concluded that the sum of the five pairs is (10)(11)/2 or 110/2 or 55. Next, she challenged the students to connect this problem to their work with the triangular numbers. After a few minutes, Curtis observed that the pattern they had first seen with the triangular numbers meant that each of these numbers was the sum of consecutive (counting) numbers. He explained that the fourth triangular number was 10, or 1 + 2 + 3 + 4, and that this also worked for all the others. Therefore, he said that 55 must be the 10th triangular number, since it was the sum of the first ten (counting) numbers. After some discussion to be sure that everyone saw the connections among the original recursive pattern, the observation made by Curtis, the work they had done in adding the first ten counting numbers, and Tamika's rule, the teacher and students together put the finishing touches on validating Tamika's generalization.

What is the teacher's role in supporting reasoning and proof in the mathematics classroom in grades 6–8?

One of the ways that teachers can help middle grades students appreciate and use the power of reasoning is to expose them to a variety of problems that require reasoning to investigate mathematical ideas. In the example below, the teacher presents students with a situation that requires a systematic approach and also teaches a valuable lesson about generalizing patterns.

The teacher asked students to determine how many different (unique) length segments could be made by connecting nails on their 5-by-5 geoboards. The students connected the nails with rubber bands to form segments that they thought were different lengths. Their first arrangements were similar to those in figure 6.10.

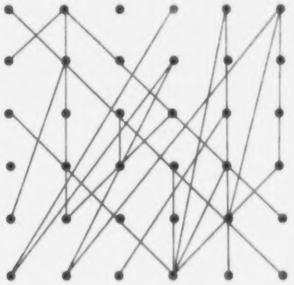


Figure 6.10. Unorgranized geoboard segments.

Because students were going to have to check for duplicate segments and could use the growth in the number of segments to establish patterns, the teacher directed the students to make arrangements that were systematic. She asked, "Do you see any patterns in the number of segments?" When they looked at their geoboards, they commented, "I can't see any patterns" or "This is just a bunch of lines." Their boards, which had little structure, caused their confusion. The teacher asked the students if the segments could be organized to help them identify patterns if they existed. The students were not clear on what steps to take. To help students to reason about the segments in a systematic way, the teacher asked them first to identify the different segments that could be connected in a 1 x 1 square. They quickly identified two different segments (see figure 6.11(a)). She asked, "How do you know that they are different lengths? The students explained that one was the side of the square and the other was longer because it was the diagonal.

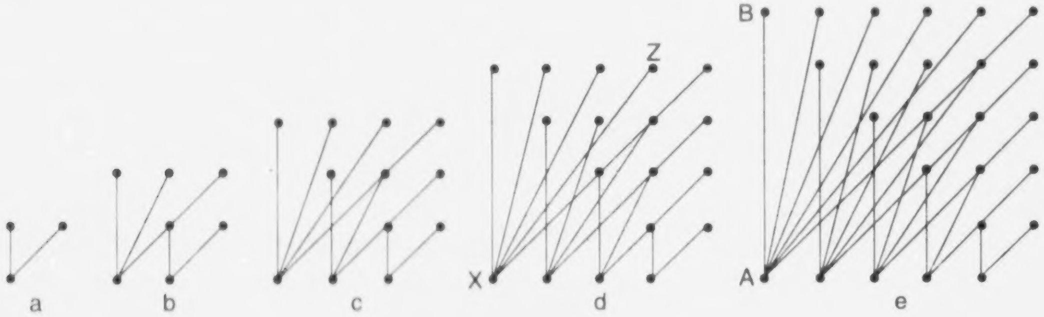


Figure 6.11. Pattern of segments on a geoboard.

Next she asked the students to consider segments on a 2 x 2 square. After most had finished this task, she asked one student to show his method to the class. He explained that he first put the two segments from the 1 x 1 square on his board and then added the new segments for the 2 x 2 square (see figure 6.11(b)). It happened that several other students had also used this approach. Although the teacher knew that there were other approaches that could be used to generate all possible lengths, no students appeared to be using them, so she decided to suggest this kind of systematic approach to the class. She asked them to try to use this approach to examine the number of segments on 3 x 3 and 4 x 4 boards. Eventually, the class generated arrangements resembling those in figures in 6.11(c-d). When students finished, the teacher asked them to describe the number of segments numerically. They counted the segments and recognized that in each successive square the number of new segments increased by 3, 4, and 5. A student said, "That's one for every nail on the side of the square!" They decided to put the information into a table (see table 6.3).

The teacher asked, "Everyone has the first four squares complete. Can you make a conjecture about the number of segments in the next square?" The students counted the number of segments using their squares and the table. Many conjectured that the 5 x 5 square would have twenty different segments (2 + 3 + 4 + 5 + 6 or 14 + 6). They connected the segments on the geoboard and saw the additional 6 segments. They thought they were finished with the first part of the exercise and could make a generalization.

Squares	Segments by Addends	Total Segments
1 x 1	2	2
2 x 2	2 + 3	5
3 x 3	2 + 3 + 4	9
4 x 4	2 + 3 + 4 + 5	14

Table 6.3. Unique segments in four geoboard squares.

The teacher reminded the students about the requirement for unique segment lengths and asked them to find the exact length of all segments. One student said, "We can measure them with rulers." Another said, "We could look at the triangles to see if we have different ones." The teacher asked for more explanation of that approach, and the student commented that each of the rubber bands not parallel to one of the sides represented the hypotenuse of a right triangle, so they could use the Pythagorean theorem to find the lengths. The students found the measures of all the lengths using both methods and found that segment AB in figure 6.11(e) is the same length as segment XZ in figure 6.11(d). They are both five units long. Thus, the number of unique segments was only 19, not 20. The pattern failed to hold, as table 6.4 indicates.

Squares	Segments by Addends	Total Segments
5 x 5	2 + 3 + 4 + 5 + 5	19

Table 6.4. Number of segments in 5x5 square.

In developing students' reasoning, teachers play an important role by asking questions to help students understand the problem, to organize data, and to make and validate conjectures. The teachers in both the episodes described above used questions, observation, and student work to help the class develop a systematic approach to reasoning about patterns. The teachers selected appropriate problems that helped students develop different approaches to reasoning about and solving problems. The choice of the problem in the geoboard example also gave the teacher a natural opportunity to discuss the difference between a conjecture and a proof. Not all patterns generalize in ways that we might wish or expect from early observations. This was an important lesson that gave her students a healthy skepticism in their future work with patterns. A teacher's guiding questions, such as those used in the figurate number and geoboard segment problems, are crucial in helping students organize data, make conjectures, and develop and evaluate arguments. Teachers are major sources of inspiration and guidance to students as they develop their mathematical reasoning.

Standard 8: Communication

Mathematics instructional programs should use communication to foster understanding of mathematics so that all students—

- **organize and consolidate their mathematical thinking to communicate with others;**
- **express mathematical ideas coherently and clearly to peers, teachers, and others;**
- **extend their mathematical knowledge by considering the thinking and strategies of others;**
- **use the language of mathematics as a precise means of mathematical expression.**

Elaboration: Grades 6–8

Communication is an essential feature in classrooms where students are challenged to think and reason about mathematics and to express the results of their thinking to others orally or in writing. This type of learning environment is desirable at any grade level, but there are a few distinctive features of such communities in the middle grades. In the middle grades, the mathematics under discussion is generally more complex and perhaps more abstract and hypothetical than the mathematics in earlier grades. A second distinctive feature relates to the norms for evaluating the thinking of members within the classroom community. When students in grades 6–8 explain their thinking, they can be held to more stringent standards than would likely be applied to younger students, though not as rigorous as might be applied in high school. Social norms are also critical in the middle grades. Thus, a third distinguishing feature of communication in the middle grades pertains to the students themselves, rather than to the content of their discussions. During adolescence, students are often reluctant to do anything that causes them to stand out from the group, and many middle school students are self-conscious and reluctant about exposing their thinking to others. Peer pressure is powerful, and a desire to fit in is paramount. Teachers need to build a sense of community in middle grades classrooms so students feel free to express their ideas honestly and openly, without fear of ridicule.

What does communication look like in grades 6–8?

Consider an example of mathematical communication in a middle grades classroom. In Mrs. Miller's classroom students often work collaboratively to solve challenging problems. One day she presented the following problem:

> The ratio of the length of a rectangle to its width is 4 to 3. Its area is 300 square inches. What are its length and width?

While the students worked in pairs, Mrs. Miller responded to students' questions in a way that kept their focus on thinking and reasoning. She resisted students' attempts to have her "do the thinking for them" by responding with suggestions (e.g., "Try to think of some way to use a diagram") or further questions (e.g., "What do you know about the relationship between the area of a rectangle and its length and width? How can you use what you know?"). She also noted different solution approaches that were used by the students and made decisions about which students she would ask to present solutions.

After most students had a chance to obtain a solution, Mrs. Miller asked Angela and Dorothy to go first. At the overhead projector, they explained their approach by first drawing a table with three columns labeled L (for length), W (for width), and A (for area). In the first row of the table, Angela wrote three numbers: 4, 3, 12. Mrs. Miller asked her why she wrote these numbers. Angela claimed that these were the smallest possible values for the length and width, but that the area was not correct. Then she explained that they had continued to list the next largest possible values for the length and width to see if they could detect a pattern. On the overhead, she listed the next several rows in their table: (8, 6, 48), (12, 9, 108), (16, 12, 192), (20, 15, 300). Angela pointed to the final row and explained that this showed that the solution was 20 and 15 (see table 6.5). At this point Mrs. Miller, who noted that Dorothy had not participated in the presentation, asked Dorothy if she could explain why they thought their solution was correct. Dorothy explained: "The area of a rectangle is length times width. If you do 20 times 15 you get 300, and that is what we needed." Darryl then asked why 30 and 10 was not a solution, since 30 x 10 is also 300. Dorothy replied that 30 and 10 could not be a correct answer because those numbers did not appear in their table. Mia then raised her hand and pointed out that the sides had to have a ratio of 4 to 3, and that 30 and 10 did not have that ratio. Dorothy then explained that this was what she meant when she said that those numbers did not appear in the table—only those pairs with the correct ratio were in the table.

Length (L)	Width (W)	Area (A)
4	3	12
8	6	48
12	9	108
16	12	192
20	15	300

Table 6.5. Plausible values for length and width.

Mrs. Miller thanked the girls for sharing their work, and she asked if she could have the overhead transparency of the table they had created for the problem. She planned to save it for a future lesson in which the students would consider the effect on the area of multiplying the length and width of a rectangle by a certain factor. She then asked if anyone had further questions for the two girls. Lee raised his hand and said that he and his partner had solved the problem in a similar way, but he thought their way was easier. Mrs. Miller invited them to present their solution next. Lee and Randy proceeded to the overhead projector to explain their work. After briefly restating the information given in the problem, Lee indicated that 3 times 4 was equal to 12, and they needed "a number that both 3 and 4 would go into." Mrs. Miller asked why they had multiplied 3 and 4, to which Randy replied that the ratio of the length to the width was given as "4 to 3" in the problem. Lee went on to say that they had determined that "3 goes into 15 five times and that 4 goes into 20 five times, too." Since 15 times 20 was equal to 300, the area of the given rectangle, they concluded that these numbers represented the width and length of the rectangle.

Mrs. Miller asked the class if they had any questions for Lee and Randy. Echoing Mrs. Miller's query during the presentation of the solution, Tyronne commented that he did not understand their solution, particularly where the 12 had come from and how it helped them solve the problem. Neither Lee nor Randy was able to explain why they had multiplied 3 and 4, nor how the result of that multiplication connected to their thinking in obtaining the solution. Mrs. Miller then indicated that she also wondered how they had obtained the 15 and 20. The boys reiterated that they had been looking for a number "that both 3 and 4 went into," after which Mrs. Miller and Darryl simultaneously asked how the boys had obtained the number 5. Lee and Randy responded that 5 was what "3 and 4 go into." At this point, Keisha asked, "You guys are a little confused. Did you just guess and check?", to which the boys responded in unison, "Yeah!" Although Lee and Randy's answer to the problem was correct, and although it contained a kernel of good mathematical insight, their explanation of their solution method left students confused. Mrs. Miller decided to ask for yet another solution.

Because she had seen Rachel and Keisha use a different solution method, Mrs. Miller asked them to present their solution. Keisha made a sketch of a rectangle, labeling the length 4 and the width 3. She explained that the 3 and 4 were not really the width and length of the rectangle, but that the numbers helped keep the relative size of these lengths in mind. Then Rachel explained that you could imagine 12 squares inside the rectangle, because 3 times 4 was equal to 12. To show this, she drew lines to subdivide the rectangle accordingly. Next she explained that the area of the rectangle must be equally distributed in the 12 "inside" squares. Therefore, they divided 300 by 12 to determine that each square would contain 25 square inches. At Mrs. Miller's suggestion, Rachel wrote a 25 in each square in the diagram in order to make this point clear. Keisha then explained that, in order to find the length and width of the original rectangle, they had to determine the length of the side of each square. Since the area of each square was 25 square inches, she argued, the side of each square was 5 inches. Then, referring to the diagram (figure 6.11), she explained that the length of the rectangle was 20 inches, since it consisted of the sides of four squares, and the width was 15 inches.

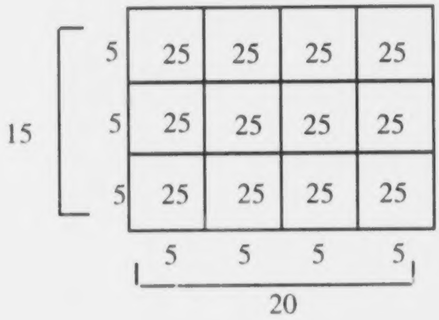


Figure 6.11. Rachel's and Keisha's method.

A few students asked questions to clarify their understanding of the solution, and these were answered well by Keisha and Rachel. Then Mrs. Miller asked the girls if they thought their approach could be used for any similar problem. For example, she asked, what if the ratio were not 4 to 3, or what if the area were not 300? Rachel responded that she was confident that the approach would work in general, but Keisha was less certain. Darryl then asked what would happen if the product of the "length/width ratio numbers did not divide evenly into the area" of the rectangle. This question generated a lively exchange among several students, during which Mia asked if the method used by Angela and Dorothy would work in such cases. Mrs. Miller was pleased that these matters of generalization were being considered by some of the students, but she wanted to be sure that she engaged the entire class in this kind of thinking. Because time was running out in the period, she asked the students to consider this issue for homework. In the final few minutes of class, she asked students to record in their journals their observations about what they learned during the lesson and what questions they had.

What is the teacher's role in supporting communication in the mathematics classroom in grades 6–8?

Mrs. Miller has established a classroom community in which communication is central to teaching and learning mathematics. Of course, not every day goes as well as this one. For several years she has worked collaboratively with her colleagues in the school to establish communication-rich mathematics instructional environments in which students are encouraged to share their ideas and to seek clarification until they understand. The focus is on trying to make sense of mathematics together as a community. As the teacher, Mrs. Miller assumes special roles, including monitoring and facilitating discussions. In her role as facilitator, Mrs. Miller tries to support students as they take substantial responsibility for their own mathematics learning and that of their peers. She also steers the "mathematical direction" of the conversation to be sure her objectives are addressed. In this classroom atmosphere of mutual trust and respect, it is acceptable to struggle with ideas, to make mistakes, and to be unsure. Students are willing to participate actively because they know that they will ***not*** be criticized personally, even if their mathematical thinking is critiqued. Explanation, questioning, debate, and sense-making are natural and expected activities in this mathematics classroom.

Communication in Mrs. Miller's classroom focuses on tasks that engage students in thinking and reasoning about something mathematically important. Although the task in this lesson was in many ways quite simple, with the teacher's purposeful guidance it provided students with an opportunity to draw on their understanding of area and ratio, an important instructional emphasis for the middle grades. Typical of "worthwhile mathematical tasks" (NCTM 1991), this task was accessible to multiple solution methods; allowed multiple representations; required students to justify, conjecture, and interpret; and related to important mathematical ideas. Tasks with these qualities provide occasions for mathematical discourse.

This task was also generative for Mrs. Miller in at least three directions. In the days immediately following this lesson, she and her class explored the use of algebraic representations and solution methods for this and related problems in their quest to find a method that would work for a larger range of possible values for the areas and the ratios of sides. Also, in a later lesson, she used the overhead transparency generated by Angela and Dorothy to have the class examine how the area of a rectangle is affected when the dimensions of a rectangle are multiplied by a certain factor. Finally, she used this task to help her students consider the nature of an adequate explanation for the solution to a mathematical problem. In order to acquaint her students with the scoring guidelines that would be used on the state proficiency test, she had them prepare written solutions for this problem and then score each other's solutions using these guidelines. In this way, they learned valuable lessons about the need for accuracy, precision, and completeness in the language used in their written communication.

In Mrs. Miller's classroom, the selection of a task that had the potential to both engage students and challenge them to think and reason was key. Using the task to facilitate student mathematics learning required her careful attention. There are numerous issues that teachers must consider in helping students learn to talk about mathematics, one of which is how to orchestrate a classroom conversation. Who speaks? When? Why? Encouraging students to speak can sometimes be in tension with advancing the mathematics. Some responses will be more relevant or more insightful than others. It is important to help students recognize good ideas. On the other hand, sometimes it is productive for the teacher to pick up on, and probe, incorrect responses. Only by bringing misconceptions and errors out for inspection can they be examined and appropriately dealt with. And attention to exceptional responses must be balanced by actions that encourage participation by all. It is important to ensure that everyone has opportunities to contribute, though it is not necessary to give equal speaking time to all students.

Mrs. Miller's voice was heard during each of the student presentations and ensuing discussions, which took about fifteen minutes overall. Her comments were crucial to the flow of discourse, but her silences during the presentations and discussion were also important. It is clear that students in the class were accustomed to being asked, not only by the teacher but also by other students, to explain their mathematical thinking and reasoning. Time is also important. Mrs. Miller gave her students sufficient time to generate and explore their own ideas and to discuss problem solutions, initially with their partners and subsequently with the entire class. Mrs. Miller and many students pressed for justification and explanation for each solution presented. This carried a dual message—thoughtful inquiry is valued, and understanding the method used to solve a problem is important.

For those students who did not participate in the discussion, other methods were in place to engage them in communication. For example, Mrs. Miller's students all had opportunities for communication as they worked in pairs. This approach is often very effective with students in the middle grades. In the relative privacy of a small group, students can try out their ideas with just one or a few people before opening themselves up to the entire class. And Mrs. Miller offered her students other opportunities for individual reflection and communication, such as journals, written commentaries, and individual homework.

Standard 9: Connections

Mathematics instructional programs should emphasize connections to foster understanding of mathematics so that all students—

- **recognize and use connections among different mathematical ideas;**
- **understand how mathematical ideas build on one another to produce a coherent whole;**
- **recognize, use, and learn about mathematics in contexts outside of mathematics.**

Elaboration: Grades 6–8

Connections are inherent in learning mathematics at all grades. Looking for connections is part of what it means to think mathematically, and making connections builds mathematical knowledge. Without connections, students must learn and remember too many isolated concepts and skills, rather than build on previously learned and understood ones. Teachers need to continually provide opportunities to help students see and communicate connections as well as build the disposition for seeking and using connections. This can be done by the questions teachers ask and can be supported by curriculum materials that foster looking for connections as part of what it means to study mathematics.

In the middle grades there are many new opportunities to use and build connections due to the additional mathematical content that students address. A particular focus is making connections between the key mathematical concepts that students develop in these grades (rational numbers, proportions, and linear relations), and how these ideas are related to other content strands.

Connections can also be made to the rich problem contexts that arise in the expanded content of other disciplines, as well as the different real world and daily life experiences that interest middle grades students. At this level tools such as graphing calculators become important in the study of mathematics. All of these serve as the bases for connections and opportunities to further students' mathematical understanding.

What do mathematical connections look like in grades 6–8?

Mathematics classes at the middle grades level should provide opportunities for students to continue to see and experience mathematics as a whole through the curriculum they receive and the questions their teachers and classmates ask. Students reveal the ways in which they are connecting ideas in answering: Why? How did you arrive at your answer? Why does that make sense? Did anyone think about this in a different way? How are the two ideas related? How does our work today relate to what we have done in earlier units of study? From these same discussions there is the potential that others may develop new connections and enhance their understanding of mathematics by hearing how classmates have addressed a situation.

With curriculum and instruction focused on mathematics as a discipline of connected ideas, students learn to expect that mathematical ideas are related. From this perspective, new ideas are seen as extensions of previously learned mathematics. Therefore, students use what they already know to address new situations. With this disposition, students should routinely ask themselves include: How is this problem like things I have done before? and How is it different?.

Rich mathematical tasks and discourse create the setting for students to use and develop mathematical understanding and connections. Tasks provide the stimulus for students to think about particular concepts and procedures, their connections with other mathematical ideas, and their applications to real-life connections. Discussions that require students to reason and communicate mathematically are more likely to promote their ability to solve problems and develop new connections.

For example, the task below was used with seventh graders. The students were studying a unit on ratios and proportions and the intent of the problem was to begin to develop their understanding of methods of comparing ratios. In the process of finding solutions, students made several connections, especially within the number strand.

> Southwestern Middle School Band is hosting a concert. The seventh-grade class is in charge of refreshments. One of the items that is to be served is punch. The school cook has given the students four different recipes, calling for sparkling water and cranberry juice.
>
> **Recipe A**
> 2 cups cranberry juice
> 3 cups water
>
> **Recipe B**
> 4 cups cranberry juice
> 8 cups water
>
> **Recipe C**
> 3 cups cranberry juice
> 5 cups water
>
> **Recipe D**
> 1 cup cranberry juice
> 4 cups water
>
> - Which recipe will make punch that has the strongest cranberry flavor? Explain your answer.
> - Which recipe will make punch that has the weakest cranberry flavor? Explain your answer.
> - The band director says that 120 cups of punch are needed. For each recipe, how much cranberry juice and how much sparkling water are needed. Explain your answer.

The students worked on the problem in groups of two or three. When students had finished, they came together as a whole class to share and explain answers. The class began by discussing the first two questions. Groups of students interpreted the question of which recipe has the most cranberry juice in a variety of ways. Some used the part-whole relationship of the number of cups of juice to the total number of cups in the recipe (2/5, 4/12, 3/8, 1/5). Others looked at the ratio of juice to water (2/3, 4/8, 3/5, 1/4). Still others looked only at the number of cans of juice in each recipe (2, 4, 3, 1). After questioning one another's solutions and comparing methods, the class moved on to the last question. Below are four of the strategies the groups used to work through this part of the problem.

> **Group with Recipe A:** We figured out that each recipe would make 5 cups: 2 of juice and 3 of water. So to make 120 cups, it would take 120 divided by 5 and that is 24, the number of recipes needed. Since we need 2 cups of juice and 3 cups of water for one recipe so, we need 2 x 24 = 48 cups of juice and 3 x 24 = 72 cups of water for the 24 recipes. And since 48 cups of juice + 72 cups of water makes 120 cups of punch that must be right.
>
> **Group with Recipe B:** We thought that since 4 cups of juice and 8 cups of water is the same ratio as 1 cup of juice and 2 cups of water, we then thought about the 120 cups of punch as divided into three groups of 40 cups each: 40 + 40 + 40 = 120. We need 1 part juice, so that is 40 cups, and 2 parts water, so that is 80 cups. This makes 120 cups of punch and you still have a ratio of 1 part juice to two parts water.
>
> **Group with Recipe C:** We tried to double the recipe, but that was not enough. So we added another batch and that still was not enough. So we just kept adding recipes and seeing how many total cups of punch we had. We kept up this pattern until we got 120 cups. So we had the following (Table 6.6).

Cups of juice	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45
Cups of water	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75
total cups	8	16	24	32	40	48	56	66	72	80	88	96	104	112	120

Table 6.6. Punch recipes chart.

> That means we had 45 cups of juice and 75 cups of water. (Later in the class discussion, this group noticed that they could have gone directly from 3/5 to 45/75 by multiplying the numerator and denominator by 15 because they needed 15 recipes.)
>
> **Group with Recipe D:** We tried various numbers. First we tried 20 cups of juice. This means we needed 4 times as much water or 80 cups of water. But this was too small because 20 + 80 is only 100. So we tried 30 cups of juice, so that meant 30 x 4 =120 cups of water. This time we had too much punch, 30 + 120 = 150. Next we tried 25 cups of juice. And 25 x 4 = 100, so we had 100 cups of water. But this made 125 cups of punch which was close but too much. So we tried 24 cups of juice, which needed 24 x 4 = 96 cups of water. This worked because 24 + 96 = 120 cups of punch.

After groups had shared their strategies, the teacher continued the conversation and had the class talk about the similarities and differences among the strategies used by each group. Some students concluded that the number of recipes needed to make 120 cups of punch could be found by finding the multiple needed to obtain an equivalent ratio:

$$\frac{\text{cups of juice}}{\text{total cups of punch}}$$

With this problem, the students experienced mathematics as the integration of several ideas: fractions, ratios, proportions, operations, magnitude, scaling, number sense, patterns, and so on. The "making punch" task is a single problem, and therefore does not necessarily offer an opportunity to make connections to all mathematics content strands. But it does present mathematics as a tool for solving problems rather than as a set of isolated facts to be memorized. Part of completing this task involved students communicating their strategies for solving the problems. Their solutions showed the numerous and different mathematical ideas being used and connected. All students had an opportunity to enhance their understanding of ratios in listening to others' ideas. Some students built from previously learned mathematics and used a sense of variable, but not all in exactly the same way. The groups used different mathematics and different strategies. For example, the group with recipe D used the strategy "guess and check" and their understanding of number to solve the problem. The group with recipe C built a table and used the idea of scaling ratios (as well as recursively adding) in the same way that students find equivalent fractions. The groups with recipes A and B thought about comparing quantities and using ratios. Yet, students in the different groups also used a sense of variable as generalized number, even though they did not use algebraic symbols. A closer examination of the reasoning process reveals that students were developing understandings that lay a foundation for later study of proportions, rates of change, and linear relationships.

What is the teacher's role in supporting students' understanding of mathematical connections in grades 6–8?

The teacher's role includes engaging students in problems that connect mathematical ideas within topics and across the curriculum, as well as getting students to build on their current mathematical ideas to develop new ideas. In the "making punch" problem, the teacher's orchestration of the lesson allowed students to make the connections explicit, and her questions about the similarities and differences among students' strategies focused on relationships and generalities. The teacher took advantage of an opportunity to foster the students' disposition to look for connections as well as to use connections. In situations like these, it is critical that the teacher recognize and understand the concepts developed, not just in order to teach the abstract manipulation, but also to be able to orchestrate the conversation. The teacher also needs to be able to make quick decisions about the next steps so as to focus students' attention on the appropriate use of words and notations that will support their understanding of new concepts, such as proportionality and algebra.

Teachers can enhance the extension and expansion of students' understanding of the discipline of mathematics by using other disciplines as sources of problem contexts. Science and social studies are rich in opportunities to learn about measurement, data, and algebra; while art and computer graphics can be used to make sense of shape, symmetry, similarity, and transformations of geometric figures. The context of environmental studies offers a venue for the study of numbers—large numbers (amount of garbage produced by a family, community, state, or province); geometry and measurement—volume, surface area, scale factors, ratios (building different sized compost boxes); data and statistics—posing questions, collecting data, analyzing the data, and interpreting the results (study of the effects of pollution on animal and plant populations); and many others.

In many middle schools, teachers are interested in fostering interdisciplinary studies. If done well, this can help students connect the mathematical ideas that are often used in other disciplines to the ideas discussed in mathematics classes. The mathematics teacher could work with teachers of other disciplines to develop integrated units of study. For example, middle grades science classes might be studying wildlife such as deer, fish, or eagles. If students are to use sampling techniques to determine the population of a species, it is important that the mathematics and science teachers talk about students' understanding of the different sampling techniques, especially the idea of randomness. It is also important that science teachers understand that students are likely to use scaling and equivalent ratios to estimate the total population rather than using cross-multiplication and formal algebraic symbol manipulation to find their solutions. These conversations about students' experiences, understandings, and procedures give the teachers the opportunity to discuss specific issues concerning the mathematics, such as algorithms and the level of abstract symbol manipulation that students may use. Without conversations of this nature, teachers may use procedures that are not part of students' repertoire, and therefore the students may miss an opportunity to expand their use of reasoning or to see that what they are doing in mathematics class can be used in other disciplines. This is not to imply that using mathematics in the study of science, social studies, or any other discipline is sufficient to count as a middle grades mathematics curriculum. Instead, interdisciplinary experiences serve as avenues along which mathematical ideas can be revisited. These types of learning experiences help students see the usefulness of mathematics and connections between what they do in mathematics class and what they study in other classes.

In this same spirit, mathematics teachers can build on and connect to disciplines other than science. For example, mathematics teachers can talk with language arts teachers about the development of communication skills as well as reading strategies. Here language arts teachers can inform mathematics teachers about the development of student writing at the middle grades level as well as strategies for improving the writing the students do in mathematics class. Again, students benefit from efforts by teachers to understand each others' disciplines and to make connections between the disciplines. With all the teachers in a building focusing on connecting content areas, mathematics and the other disciplines are seen as permeating life and not just as isolated topics of study.

Connections cut across the process strands, as demonstrated in "making punch" and other examples. Problem solving and reasoning are the catalysts for connecting ideas. Students should be using communication techniques to demonstrate their understanding of connections. Also, mathematical representation needs to be connected to mathematical concepts in order for students to build mathematical knowledge. Thus, looking for connections is not only a way to study mathematics but a way to encourage students to think about, use, and engage in problem solving with mathematics.

Standard 10: Representation

Mathematics instructional programs should emphasize mathematical representations to foster understanding of mathematics so that all students—

- **create and use representations to organize, record, and communicate mathematical ideas;**
- **develop a repertoire of mathematical representations that can be used purposefully, flexibly, and appropriately;**
- **use representations to model and interpret physical, social, and mathematical phenomena.**

Elaboration: Grades 6–8

Use representations to model and interpret physical, social, and mathematical phenomena Representations—such as physical objects, drawings, charts, graphs, and symbols—are central to the study of mathematics. They help students make sense of mathematical concepts and relationships. They also provide students with a means for communicating about their thinking. Students' understanding is deepened when they themselves engage in creating, comparing, and elaborating on various representations.

At the middle school level, where fractions, decimals, percents, the relationships among them, and proportional reasoning receive major attention, students learn to recognize, compare, contrast, and work with these ideas by exploring a variety of representations for them. Another major thrust of middle school mathematics is the development of algebraic thinking, with a particular focus on linear functions. As they model and solve problems that arise in the real and mathematical worlds, middle school students learn to use variables flexibly to represent unknowns, and also to employ equations, tables, and graphs for representing and analyzing relationships.

What does representation look like in grades 6–8?

One of the major goals of the middle grades is for students to become fluent and flexible in their understanding and use of fractions, decimals, and percents and the relationships among them. For younger students, the world of fractions and the world of decimals often seem quite distinct. Even for older students and adults, there is a tendency to think of fractions as sets or regions (e.g., two-thirds of something) but of decimals as more like numbers. Actually, fractions, decimals, and percents are three different representations that can be used interchangeably to describe certain numbers. For example, 0.75, 3/4, and 75% are all equal, but simply written using different notation. Although fractions can be expressed as decimals or as percents, the connotations that these notations carry are rather different. An important task in the middle grades is to help students see how these different representations are alike and different, to identify their strengths and weaknesses for particular situations, and to develop some sense for when each is most appropriately used. As shown in Figure

6.14, fractions, decimals, and percents can be contrasted not only in terms of notation, but also in terms of the corresponding pictures or models.

Recognizing how all of these ideas are connected is important and ultimately very useful. In some contexts it is easier to think about 1/4 than about 25/100 or 0.25 (for example, when items are on sale for 25% off, we might compute one-fourth of the price mentally to determine the savings). In other situations, a decimal might be preferred to a fraction—for example, when computing with a calculator or measuring with a digital scale. On the other hand, it is more convenient to compute with fractions when working with measuring cups in cooking or hours and minutes when measuring time. Conceptual and practical advantages exist for each representation. Students can use familiar fraction concepts and pictorial models to think about operations with those rational numbers that are easily represented by decimals: tenths, hundredths, and thousandths. Students can extend the base-ten decimal system to include numbers less than 1 as well as large numbers by working, for example, with base-ten blocks where the large cube is assigned a value of 1, so that the flat is 0.1, the long is 0.01, and the small centimeter-cube units represent .001. And students can use pictorial models to make meaningful translations among fractions, decimals, and percents.

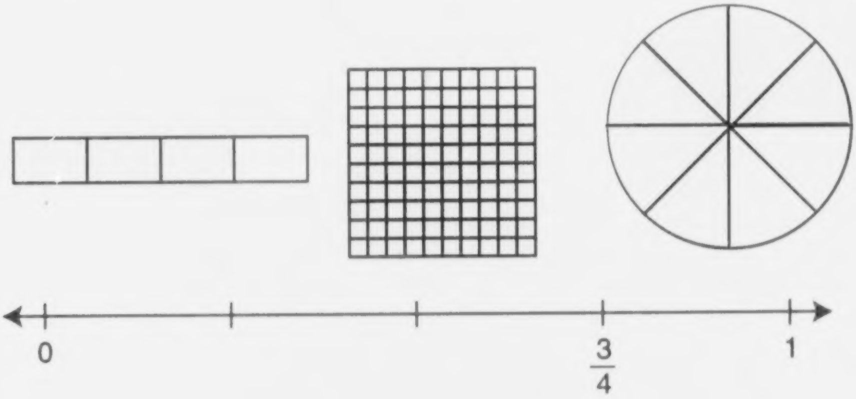


Figure 6.13. Ways to represent 3/4.

Ideas related to ratio and proportion also receive significant attention in the middle grades. Whereas problems in the elementary school frequently require students to compare quantities by finding their difference, comparisons requiring recognition of multiplicative relationships are a primary focus in the middle grades. Proportional reasoning "is of such great importance that it merits whatever time and effort must be expended to assure its careful development. Students need to see many problem situations that can be modeled and then solved through proportional reasoning" (NCTM 1989, p. 82). Representations that help students develop proportional reasoning include making scale drawings or models of figures or objects (to connect the geometric notion of similarity with the numeric concept of ratio) or drawing graphs of data taken from situations where proportionality is involved. Figure 6.13 is a graph of ordered pairs representing the length and width of similar rectangles. The graph helps students see connections among similarity, ratio, linearity, and slope.

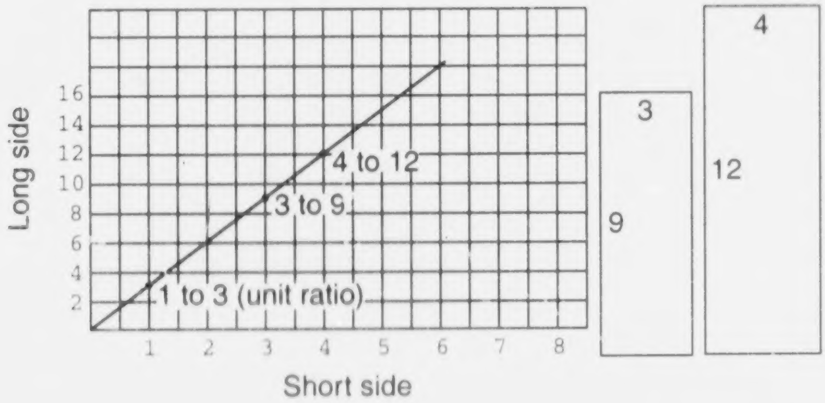


Figure 6.13. Ratio of sides in similar rectangles.

Middle grades students also use representation as a process for developing important ideas about algebra. Major learning goals for algebra at this grade-band include understanding the concepts of variable and equation; developing fluency with equivalent algebraic representations; using linear equations and other representations as modeling and problem-solving tools; becoming proficient in representing situations algebraically and in solving linear equations and inequalities; and using algebra-based reasoning to solve problems and justify conclusions. Students should be expected to relate representations of problem situations verbally (e.g., story), numerically (e.g., table), symbolically (e.g., variable expression), and geometrically (e.g., graph). It is in the middle grades that students should become fluent in representing relations and functions in these various modes and in choosing appropriately among them in order to solve problems.

For most students, extensive work with symbols should come only after they have become fluent with verbal, tabular, and graphical representations of the relationships that symbols can also represent. Teachers find it helpful to introduce work with symbols in conjunction with other forms of representation that may be more familiar to students. For example, students might explore problems like the following:

> A rectangular swimming pool (L by W) is to be surrounded by a ceramic tile border. (Assume that L and W are natural numbers.) The tile border will be one tile wide all around. Explain in words, and in symbols, the number of tiles (T) that will be needed.

Some students might use a table to record values for L, W, and T for rectangles of various sizes; others might reason more geometrically or more abstractly about the situation. Here are four possible student responses:

> I drew a picture. You need L+2 tiles across the top, and the same number across the bottom. Then you still need W tiles on the left and W tiles on the right. So altogether, the number of tiles needed is $T = 2(L + 2) + 2W$.

> The formula for the number of tiles is $T = 2(L + W) + 4$. I made a table with columns for L, W, and T. I drew some pictures. Then I counted the tiles for those pictures, filled in the numbers in the table, and looked for a pattern. It's easy! You always just add the length to the width, double that answer, and then add four.

> I pictured it in my head. First, place one tile at each of the corners of the pool. Then you just need L tiles across the top and the bottom, and W tiles along each of the sides. So altogether, the number of tiles needed is $4 + 2L + 2W$.
>
> You can find the number of tiles needed by finding the area of the whole rectangle (pool plus tile border), and then subtracting off the area of the pool. The area of the pool and deck together is $(L + 2)(W + 2)$. The area of the pool alone is LW. So altogether, the number of tiles needed is $(L + 2)(W + 2) - LW$.

By working on problems like the swimming pool problem, students develop experience with relating symbolic representations of situations to other representations (e.g., diagrams and tables). They also see that different symbolic expressions can all represent the same situation, which leads to the idea of equivalent expressions. The class can now spend time talking and thinking about why the four expressions obtained by the students must be equal, a step toward developing meaning-based rules for symbol manipulation. For example, students might observe, by reference to their sketches, that adding two lengths to two widths ($2L + 2W$) is actually the same as adding the length and width twice $2(L + W)$, and thus recognize a pictorial representation for the distributive property of multiplication over addition—a useful tool in rewriting variable expressions and solving equations.

Using representations to organize and record data helps students make sense of complex problems and enables them to find patterns. A seventh-grade class might work on a problem like the following:

> A creative writing club got two bids on the cost of printing books. Chris's Print Shop charges \$100 to set up the presses plus \$4 for each book printed. Kelly's Book Nook charges \$25 to set up the presses plus \$7 for each book printed. How do the costs vary according to number of books printed? Who has the best prices?

The first step in analyzing any set of data is to organize and record the data, using some form of representation such as a table, chart, graph, or equation. (This problem can be approached with just paper and pencil, although use of a graphing calculator expands students' options considerably.) Some students might begin by making a table, such as table 6.7.

# of books	0	1	2	3			
Chris's Price	\$100	\$104	\$108	\$112			
Kelly's Price	\$25	\$32	\$39	\$46			

Table 6.7. Prices of books.

Other students might plot ordered pairs on the coordinate plane, using different symbols (perhaps * and #) to represent Chris's points and Kelly's points, respectively. Some students might represent the pattern verbally by an expression such as "Chris's price is \$100 + \$4 per book," or develop their own form of symbolic representation (such as CP = \$100 + 4 x B). Some of these representations are more helpful than others in answering questions that may arise, such as: How much would each shop charge for 10 books? For 20 books? How easy is it, from your representation of the problem, to find the cost for any number of books? Which shop is cheaper if you have a very small order? A very large order? Is there any number of books that would cost the same at both shops? Can you get more books from Chris or from Kelly if you want to place the largest possible order that will cost less than $300?

As students become comfortable using a variety of representations, they will find that representations are an important tool in communicating their thinking to others. Small groups of students might talk about how to represent the book problem by graphing two equations simultaneously on the coordinate plane using graphing calculators. Others might compare tables of numbers. When the whole class is brought together, the students should compare their graphs, tables, or equations. Different groups might explain their thinking. Important questions may arise: Why did you decide to use a scale of 5 for the y-axis? Does the graph look different if you use a different scale? What would happen to the graph if Chris's price per book increased from 4 to 6?

One of the strengths of mathematics is that the same, or very similar, representations are useful for many different types of problems. Suppose the students who had worked on the book printing problem are later given this problem:

> You have $23 already saved from allowance and gifts. You can save $12 dollars every two weeks. How many weeks will it take you to save $89 dollars for a new pair of athletic shoes? How long would it take to save $279 for a mountain bike? How much could you save in a year? Explain why you chose the representations you did.

After calculating some values, or making a table or graph, students should notice that the essential mathematical features of this problem are the same as the book-printing problem. Both problems involve linear functions. The power of this realization is that students can use the same types of representations and approaches to solve this problem that they used in the former problem. All the techniques that apply to studying linear equations can be brought to bear on this new problem.

Finally, it is important that middle grades students have opportunities to put their repertoire of mathematical representations to use in solving relatively large-scale, motivating, and significant problems. The goal of this sort of mathematical modeling is for students to develop a deeper understanding of the uses of the mathematics they know and to gain a greater knowledge and understanding of their world. For example, students might decide to investigate problems associated with trash disposal and recycling by collecting data on the volume of paper discarded in their classroom or home over a period of weeks or months. After organizing their data using graphs, tables, or charts, the students should think about which representations are useful in illuminating regularities in the data. From those observations, the students might then be able to offer thoughtfully justified estimates of the volume and types of paper discarded in their entire school, school district, or city in a week, month, or year, and perhaps even to make recommendations for reducing the flow of paper into landfills and trash incinerators.

What is the teacher's role in supporting students' understanding of mathematical representation in grades 6–8?

The teacher's role is to help students learn to use representations flexibly and thoughtfully in their mathematical investigations. It is important to support students as they experiment, in unfamiliar problem situations, with their own idiosyncratic, unconventional representations. Listening, questioning, and making a sincere effort to understand what students are trying to communicate with their drawings or writings is essential. For example, middle grades students need many experiences to develop a robust understanding of the very complex notion of variable. Along the way, they may use letters to stand for numbers in unconventional ways. Teachers need to be careful about trying to move students too quickly toward conventional representations. If what the teacher proposes is too far from what the student was thinking, the suggestion may be counterproductive. For example, students may be confused by the use of symbols to represent variable unknown quantities (e.g., $y = kx$, where k is considered a constant, yet can vary, and both x and y can also vary). When they first work with variables, students tend to think of them as representations only for unknown fixed constants (e.g., the solution to an equation such as $4k = 16$). Teachers may need to use informal representations or students' invented representations, such as using pictures of cups to represent the number of cups in a stack, to build connections between the students' thinking and conventional representations.

Encouraging students to talk with one another about the graphs, pictures, or symbols that they have used is also key. When students see how others interpret what they have written, and have opportunities to see how others have represented the same ideas, they may learn to analyze and critique representations thoughtfully and to recognize the characteristics that make a representation flexible, appropriate, and useful. Eventually students must learn to use conventional notation so they can communicate effectively with others. Most conventional representations are useful because of their simplicity and effectiveness in helping to solve problems. It is the teacher's job to help students make sense of the power and important features of conventional representations.

For example, a middle grades teacher interested in helping students become familiar with notions of scaling and proportionality might begin class with a whole-class demonstration using a computer or graphing calculator. He might enter a formula (for example, $y = 5x$) and ask the students to predict what the graph will look like. Then he might change the scale on the y-axis from 1 unit to 10 and ask the students to predict what the new graph will look like. He might ask what could be done to the equation to increase or decrease the slope of the line. Students should come to see that changing the scale will make the graph look different and also that keeping the scale the same but changing the coefficient will produce a different looking graph. The graphs of several different equations might be compared simultaneously. Students might be asked what a table of values might look like. Then, the students, on their own but with the use of technology, could experiment with other equations, tables, and graphs and record their thinking. This activity focusing on representation should help students better understand the concepts of proportionality and scaling.

Teachers need to be aware that the mere use of a representation by the student does not necessarily mean that the student has a better, more meaningful understanding of the problem or concept at hand. For example, requiring students to include figures or graphs in a report of a modeling project does not guarantee that students have used them in meaningful ways. Careful teacher questioning can reveal such weaknesses. For example, it is important to ask questions such as, "What does the variable k stand for? What did you learn by making the table?" "What does the graph show us?" "How did the equation help in solving the problem?" "Why did you choose to use a table rather than a graph?" "How do you think your solution would have been different if you had approached the problem graphically?" There needs to be a balance between the use of many representations and the need for meaningful use of those representations. Representation is a process through which mathematical ideas are illuminated, not a goal in and of itself.

Chapter 7
Standards for Grades 9–12

For some students, the mathematics learned in secondary school is the culmination of their formal mathematical preparation for life and for work. For others, it is the platform on which further formal study will build, in college and beyond. Either way, it is important to ask what kinds of mathematical skills and understandings students will need when they leave high school so that they are prepared to pursue a broad range of career and educational objectives. The answers to this question should suggest goals for 9-12 mathematics instruction.

Content and Process Overview

Mathematics in secondary school extends, formalizes, and builds upon the mathematics learned in grades pre-K-8. The basic groundwork has been established in most areas of content and need not be revisited. For example, in number and operation, familiarity with the whole numbers and rational numbers is assumed, and students in grades 9–12 should progress to a study of the real number system, complex numbers, and matrices. In algebra, students enter secondary school with experience perceiving and symbolizing patterns and operating on and with symbols. In grades 9–12, they should engage more deeply with formal systems and build facility with symbolization for purposes of modeling. They will learn about and use a wider range of classes of functions, including trigonometric and exponential functions. In geometry, measurement, data analysis, and probability, students solidify and expand upon what they had learned in earlier grades, and also develop new knowledge of content.

Students also should develop increased sophistication with mathematical processes, especially problem solving, representation, and reasoning. Connections across content areas and with areas outside of mathematics can become increasingly rich as students' mathematical knowledge grows. Students should engage with complex problems that call for knowledge drawn from multiple areas of mathematics and to which various solution strategies might apply. Students' reasoning and communication skills also become much more sophisticated. In high school, students learn to develop coherent, careful chains of reasoning and to make their arguments clearly and logically, both orally and in writing. These arguments should make correct use of mathematical language and representations. Where appropriate, students should be able to produce formal mathematical justifications for claims they make. Graduates of high school mathematics programs must have the high degree of mathematical literacy needed to function effectively in society.

High school students continue to develop disposition and ability to reason about mathematics. They should learn to use technology to develop deeper mathematical understanding. The key to mathematics education in grades 9–12 is the development of robust, connected understanding of algebraic, geometric, and statistical ideas.

Mathematical Needs of All Students

Regardless of their career intentions, high school graduates in the 21st century will be entering a world with very different mathematical requirements than did graduates in the latter part of the 20th century. And, they will be facing rapid change. Change will come from at least two sources.

First, technology and the acceleration toward an "information society" will require of all citizens a degree of fluency with technological tools and the capacity to manage large amounts of information and data in ways that were unimaginable just a few years ago. Whether the issue is making wise decisions as a consumer (e.g., choosing the best cellular phone plan or exploring mortgage options) or routine job skills (e.g., managing databases, making forecasts, or retrieving data from the World Wide Web), our graduates must leave secondary school with mathematical knowledge and technological sophistication that will serve them well. Moreover, as outlined in the Technology Principle, all students should be able to learn different parts of mathematics by using technological tools to represent and analyze data, create graphs, and manipulate and analyze symbolic expressions in ways hitherto inaccessible to all but the most advanced students. Such technology should be routinely accessible to students and should be exploited wisely to help students develop deeper mathematical understandings.

Second, demands for reasoning and problem-solving skills will increase in all walks of life. People are changing jobs with increasing frequency and such change requires flexibility and a wide range of skills. Even those remaining in the same job are increasingly expected to advance and update their skills to keep pace with changes in their field. Survival and success in the work force will demand that people be flexible, knowledgeable, and resourceful thinkers. The same, of course, will be demanded of those who pursue careers that depend directly on the mathematics learned in postsecondary institutions and beyond.

In sum, all students in grades 9–12 must learn to deal with complex problems that involve multiple aspects of mathematics. They must be able to analyze and solve problems that they encounter, without knowing *a priori* which areas of mathematics apply, because neither in the real world nor in abstract mathematics do problems come cleanly labeled as requiring the methods of algebra or geometry. Indeed, most meaningful problems require drawing upon multiple areas of mathematics for their solutions and can be approached from various directions. When working on complex problems, either alone or collaboratively, students must be able to apply varied subject matter knowledge and varied reasoning and problem solving skills, in integrated fashion.

Students are unlikely to develop reasoning and problem-solving skills unless they practice them—which means that curricula must afford students the opportunities to become engaged with complex problems that are rich and varied enough to encourage the development of those skills. Curricular approaches in which, for example, students study algebra completely apart from geometry and geometry completely apart from measurement or data analysis are unlikely to be effective vehicles for helping students develop the kinds of connections they need. Curricula need to help students create links between different areas of mathematics and highlight the use of mathematics in mathematical and real-life contexts.

Differentiated Curricula (Tracking)

Because the mathematical needs of all students will continue to increase in the coming years, all students need to be provided with a solid core of mathematics on which they can build, no matter what their current prospects may appear to be. Every student should take a minimum of three years of high school mathematics courses that include substantial mathematics drawn from the areas of algebra and functions, geometry, data analysis, and discrete mathematics. Number and measurement also should be addressed, but their role is less explicit in grades 9–12. The Standards and discussion in the following sections attempt to elaborate a solid core of mathematics.

As we cannot predict with any certainty what mathematics a particular student may need in the future, the focus should be on providing all students with a core of mathematical knowledge and understanding. One can easily imagine that some students might go more *deeply* into particular topics or areas of mathematics. It is time to reject the notion of *acceleration* through a shallow curriculum as the appropriate path for those students with special interest and inclination for mathematics and mathematics-related areas of study, and to focus instead on learning with greater depth. Furthermore, there is no justification for the practice of having one curriculum path for "college-intending" students and a series of dead-end tracks for those students who stray from that path.

All students should be encouraged to study mathematics every year they are in school. Curricular organizations should accommodate students' interests, intended careers, and future plans. For example, students planning to go into business or the social sciences might pursue work in statistics and data collection beyond the core proposed here; extended attention to discrete mathematics might be appropriate for students planning computer-related careers. The fourth year might be differentiated for students, depending on their future plans. At this time, students' interest may be more focused. For students intending to pursue a career in mathematics or a related field, a fourth year of mathematics could provide more formal study of mathematics than is needed by the general population.

In general, variations in the high school mathematics program, even if based on interest, should be relatively minor and should not detract from the goal of providing all students with a substantial, high-quality mathematical preparation for whatever paths they may pursue in the future.

Articulation with Postsecondary Mathematics

The core mathematical content described in this chapter should permit many more students to study mathematics at the postsecondary level, should they so desire at some point in their careers. The goal is to provide a deeper and more thorough preparation for *all* students, including those who will go on to study more mathematics.

It should be noted that although calculus was once *the* first college mathematics course, calculus at the postsecondary level is currently in transition. Many of the calculus courses being offered at the college level now make extensive uses of technology. And, whether or not they are technologically grounded, many postsecondary calculus courses are making deeper conceptual demands on students in an effort to consider fewer topics more thoroughly. While the degree to which calculus reform has taken hold at the postsecondary level is uncertain, calculus is but one of several options for students entering technical and scientific fields.

It is essential for the high school mathematics curriculum to provide adequate preparation for students who wish to study college mathematics. As indicated above, the mathematics curriculum for all students should provide appropriate background and experience with computational tools in part to help those students who will enter highly technological environments at the college level. At the same time, the mathematics curriculum should help students develop an appropriate level of procedural fluency based on understanding, so that they can call upon their own mental tools when needed. Most importantly, the mathematics curriculum must be conceptually rich, so that students who enter postsecondary mathematics courses are prepared to deal with significant mathematical ideas.

#9

Reader Reaction

We are interested in your reaction to the style and tone of Chapter 7.

- **If you are a teacher, how might it be more helpful? If you are a mathematician, how might this be improved? Which sections are most effective?**

The chapter uses examples primarily to indicate the kind of mathematical tasks students should be able to do.

- **Is this a helpful approach? Would other approaches be more useful?**
- **Would additional examples be useful? Of what kind? Suggestions?**

The chapter describes a common mathematical core for all students.

- **Is that core reasonable and realistic?**
- **Are there areas that should be emphasized more? Less?**

Standard 1: Number and Operation

Mathematics instructional programs should foster the development of number and operation sense so that all students—

- **understand numbers, ways of representing numbers, relationships among numbers, and number systems;**
- **understand the meaning of operations and how they relate to each other;**
- **use computational tools and strategies fluently and estimate appropriately.**

Elaboration: Grades 9–12

Middle grades students should have developed a solid understanding of integers and rational numbers and their properties, as well as fluency in computation, and they will have begun work with irrational numbers. They should be able to recognize and use properties of integers and rational numbers but may not have had much opportunity to contrast the properties of different systems. During grades 9–12, student understanding of systems of numbers should be enhanced through work with complex numbers and matrices—systems with properties that differ from those of real numbers. As students' knowledge of number extends into new domains, they begin to understand the common structure among various number systems as well as their contrasting features. Their increased ability to reflect on and monitor their work with numbers should lead to greater abstraction and a capability for generalization. Students in grades 9–12 also should meet new mathematical ideas. For example, their study of sequences and series opens new ways to analyze patterns and number relationships.

On entry into grade 9, students should have some understanding of counting techniques that are useful in combinatorial problems. They should have had some experience analyzing and comparing algorithms for computation, and they should have developed strategies for estimation and mental arithmetic. During high school, students should expand their knowledge of permutations and combinations by considering cases in which direct application of permutation and combination formulas is not sufficient. Students' work with computation should include attention to numerical errors in technology-generated computations. Computing technology brings the issue of approximation to the fore. Students need to distinguish and choose between exact and approximate answers in a variety of problem settings.

Focus Areas for Grades 9–12

- **Understand numbers, ways of representing numbers, relationships among numbers, and number systems**

  In grades 9–12, all students should—

  - increase their understanding of systems for representing numbers and quantities, including matrix representations for arrays of quantities;

  - compare and contrast properties of numbers and number systems;
  - begin to understand complex numbers as a superset of the real numbers and as a system containing solutions for equations that are not solvable over the real numbers;
  - become familiar with finite sequences and series, including arithmetic and geometric examples, and develop an informal understanding of some infinite sequences and series, especially geometric series.

- **Understand the meaning of operations and how they relate to each other**

  In grades 9–12, all students should—

  - develop an understanding of the meaning of and representations for operations on vectors and matrices and, with appropriate technology, be able to use these operations to solve systems of linear equations;
  - develop fluency operating on real and complex numbers, vectors, and matrices, using by-hand operations for simple cases and using technology for more complex cases;
  - continue to develop an understanding of permutations and combinations as counting techniques in increasingly complex situations.

- **Use computational tools and strategies fluently and estimate appropriately**

  In grades 9–12, all students should—

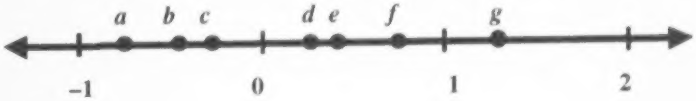
  - analyze algorithms for operations with numbers, recognize some of the roles and limitations of particular algorithms, and be able to verify the viability of selected algorithms;
  - develop an understanding of the effects of measurement error on computed values;
  - develop the ability to distinguish between estimation and approximation and use each appropriately in technological and non-technological settings.

Discussion

- **Number concepts and their properties**

Students' work with numbers and their properties in grades 9–12 should aim toward development of the ability to reflect on the relationships among numbers. By analyzing number relationships based on the position of numbers on a number line, as in the example in figure 7.1, students can develop abilities to abstract and to generalize.

Given the points as ordered on the number line below,



- Determine which point is closest to ef? To $\frac{1}{f}$?
- Decide whether the following are true, false, or indeterminable:
  - $a > b$
  - $ab > de$
  - $\sqrt{bc}$ is imaginary
  - $|c + 1| < |fg|$

Figure 7.1. Reasoning without exact values.

Students' knowledge of natural numbers can be used in problems from elementary number theory. In grades 9–12, students should formulate and prove simple number properties. For example, by building on earlier experience in exploring number patterns and relationships, students should be able to explain why the set of even numbers is closed under addition and the set of odd numbers is not. Although they might convince themselves of this using tiles or sketches, they should be able to provide an argument using algebraic symbols as well. Similarly, they should be able to argue convincingly that any prime number greater than 3 is either 1 more or 1 less than a multiple of 6, with an argument like the following: "All natural numbers are of one of the forms: $6n$, $6n + 1$, $6n + 2$, $6n + 3$, $6n + 4$, or $6n + 5$ (where n is an integer). All but $6n + 1$ and $6n + 5$ are divisible by either 2 or 3 or both. Since $6n + 1$ and $6n + 5$ are 1 more and 1 less than a multiple of 6, the only possible candidates for primes (other than 2 and 3) are numbers which are either 1 more or 1 less than a multiple of 6." They should be able to examine some of the properties of special sets of numbers, like Pythagorean triples and figurate numbers.

While an over-emphasis on the language of abstract algebra is not necessary, students can begin to appreciate mathematical systems by comparing and contrasting different sets of numbers. For example, students can note that the whole numbers are closed under addition and multiplication, the integers are also closed under subtraction, the non-zero rationals are also closed under division, and so on.

As students work with results that apply to all natural numbers, teachers can introduce mathematical induction. It is important, however, that students encounter mathematical induction at a level of abstraction appropriate to their thinking. For example, formulas for sums of arithmetic and geometric series and sums of kth powers of the first n natural numbers can be understood using mathematical induction, but the focus should remain on understanding the process rather than the production of proofs in a required format. Students can use the concept of mathematical induction in their investigations of recursive phenomena like Pascal's triangle, Fibonacci numbers, and compound interest.

With a solid foundation of experiences with equation solving (especially linear equations) in grades 6–8, students in high school can explore conditions under which polynomial functions have roots. This search can motivate the need for complex numbers. As students solve quadratic equations using symbolic manipulation calculators or programs, for example, the display of imaginary roots that will result could lead to a classroom discussion of the meaning of these numbers. Students should understand complex numbers as a system that allows solution of a larger set of equations.

Students should encounter infinite sequences and series in their analyses of infinite repeating decimals. For example, after having developed a formula for the sum of an infinite geometric series, students should be able to use it to justify that $0.\overline{9} = 1$.

As students extend their study of irrational numbers, they should develop a deeper appreciation of the distinction between rational and irrational numbers. For example, students should come to realize that there are many more irrationals than π, e, and $\sqrt{2}$, and that the irrationals in fact far outnumber the rationals.

- **Operations and their properties**

As students work with matrices, they will encounter a system in which some of the properties of real and complex numbers do not hold. Using a symbolic manipulation calculator or computer program, or computing by hand, students can test the properties of matrices. By examining several examples of matrix multiplication, they can come to appreciate matrices as a non-commutative system. By considering matrices as rules for transformations of the plane, students gain a visual model for thinking about matrix operations. Students can then use this powerful connection between algebra and geometry to gain understanding of why matrix multiplication is not commutative.

Students should also appreciate the algebra-geometry connection that the geometric representation of complex numbers affords. The diagrams and data in figure 7.2 show the product of two complex numbers, $4 + 3i$, and $1 + 2i$, which suggests a connection between complex numbers and vectors. Students might study such an example to examine the relationships between the magnitudes of the vectors representing the two factors and the magnitude of the product, or among the angles involved. Simultaneous exploration of both complex numbers and vectors helps to highlight their interrelationship and can enrich students' understanding of both.

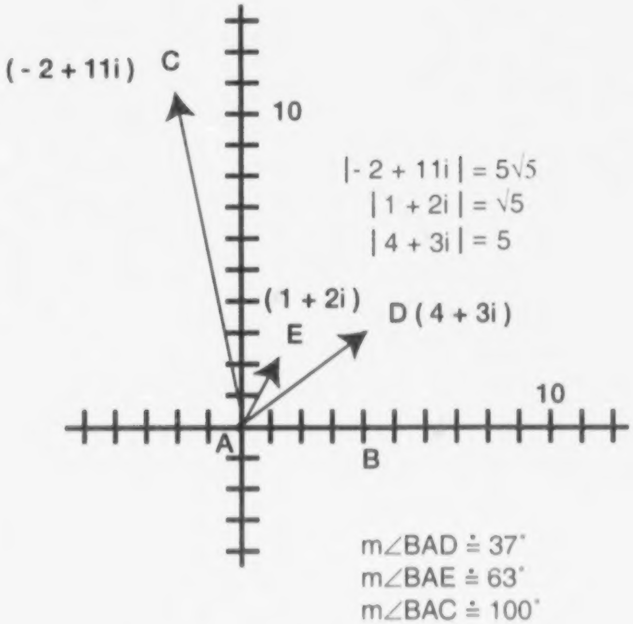


Figure 7.2. Product of two complex numbers.

Students can use technological tools to develop intuitions about the meaning of products and sums of complex numbers and the vectors that represent them. They can generate and collect data on a broad range of examples, and they can verify their geometric observations through investigation of the symbolic representations. The ease of generation of examples gives students the confidence to pursue more formal verification. As students develop a familiarity with the complex number system, they should understand the implications of the real numbers as a subset of the complex numbers.

Students' work with combinations and permutations in grades 9–12 should expand their understanding of methods of counting to include more complicated cases. For example, they should be able to analyze the mathematics of games of chance, understand a range of arguments about the probability of a DNA match, and reason about probabilistic statements made about tests for a disease.

- **Computation and estimation**

Students' analyses of algorithms should become increasingly more sophisticated than the analyses that they conducted during the middle grades. For example, based on knowledge of a division algorithm, students ought to be able to understand why any rational number can be expressed either as a terminating decimal or as an infinite repeating decimal. (See the Reasoning and Proof section in this chapter for further discussion.)

Students should develop a fuller appreciation of the significance of measurement error when studying its propagation in computation. In light of new technologies, the analysis of the effects of round-off errors and pixel size on computational and graphical results is important. Students should have a sense of what sorts of things a calculator or computer does when displaying graphs. For example, if students are exploring graphs of functions of the form $y = \sin(kx)$, they may notice that increasing the value of k causes the period of the sine curve—the distance along the x-axis needed to complete one "cycle"—to decrease. Yet if k becomes large enough, as in figure 7.3, the graph no longer appears to have the same kind of period. Additional discussion of this discrepancy can lead to deeper understanding of the concepts of periodicity and oscillation.

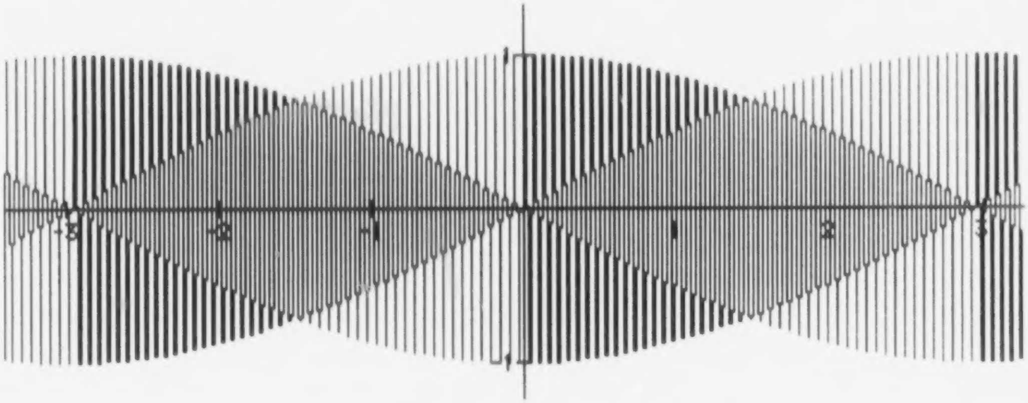


Figure 7.3. A computer-generated graph of $y = \sin(100x)$

The concepts and skills for grades 9-12 in the area of number and operation represent new mathematical content with strong connections to students' prior understandings. As students in grades 9–12 work with and reflect on new mathematical systems, procedures, and methods, they learn more about the systems with which they are already familiar.

Standard 2: Patterns, Functions, and Algebra

Mathematics instructional programs should include attention to patterns, functions, symbols, and models so that all students—

- **understand various types of patterns and functional relationships;**
- **use symbolic forms to represent and analyze mathematical situations and structures;**
- **use mathematical models and analyze change in both real and abstract contexts.**

Elaboration: Grades 9–12

A fundamental aspect of school mathematics in grades 9–12 is that it provides students with versatile and powerful tools for understanding their world while giving them access to the strength and beauty of mathematical structure. In secondary school, students should develop and expand their abilities to describe and reason about a variety of contexts using the mathematical constructs of function and relation and the symbolic language of algebra.

Students should arrive at high school with experience creating and using symbolic and graphical representations of patterns, especially those tied to arithmetic and geometric growth. They should have examined the strengths and weaknesses of various representations (verbal, numerical, symbolic, and visual) for answering questions about a situation being modeled. In grades 9–12, students' repertoire of functions expands to include periodic, piecewise, and other non-linear functions, as well as functions defined recursively. Students also use functions to define curves parametrically. Students' facility with functions becomes increasingly more flexible and sophisticated as they learn to reason about new functions derived from familiar ones via composition, inversion, linear transformation, and arithmetic combination. For example, knowing that $f(x) = \sin(x)$ oscillates with fixed amplitude, and that $g(x) = e^{-x}$ is positive and decreasing, a student should reason that the product $g(x)f(x) = e^{-x}\sin(x)$ produces a damped sine wave. Students should also develop initial familiarity with verbal, graphical, numerical, and symbolic representations of functions of two variables.

Algebra in grades 9–12 includes both the study of relationships among quantities and the study of structure. At times, each might be studied separately, but powerful mathematics can be attained when these perspectives are integrated. The issue is one of balance. Simplifying expressions and solving equations and inequalities are part of doing high school algebra. Done in isolation, these manipulations might leave the students with a sense that algebra is decontextualized symbol manipulation. In solving problems that arise from real settings, the symbolic language of algebra can be effective in finding and interpreting a solution. Students should leave high school with a balanced understanding of functions and algebra. This includes a robust understanding of the forms and properties of a broad array of functions and the ability to use these forms and properties to solve problems in a widening range of contexts. Students should understand the differences in information provided by various representations and be able to reason between and among these representations.

Students should develop an understanding of how other disciplines use algebra to model real phenomena. Teachers should understand how contextualizing mathematics through real models can help some students better understand algebra. As students move from the middle grades to the secondary grades, they move from primarily modeling linear phenomena which have constant rates of change, such as a car moving at a constant velocity, to modeling non-linear phenomena which have variable rates of change, such as the increasing velocity of a falling object or populations with exponentially increasing growth. An understanding of the relationships between non-linear functions, how those functions change, and how that change is reflected in the graphs of the functions is not only necessary for interpreting models but is also a precursor to learning calculus.

Focus Areas for Grades 9–12

- **Understand various types of patterns and functional relationships**

  In grades 9–12, all students should—

  - recognize equivalent forms of an expression, equation, function or relation;
  - be familiar with classes of functions, including linear, quadratic, power, polynomial, rational, absolute value, exponential, logarithmic, trigonometric, and step functions; understand piecewise-defined functions and their properties; analyze the effects of parameter changes; and describe local and global behavior;
  - select appropriate representations (numerical, graphical, verbal, and symbolic) for the functions and relations embedded in quantitative situations, convert flexibly among representations, interpret representations, and use them to interpret the situations represented;
  - use a variety of symbolic representations, including recursive definitions and parametric equations, to explore the behavior of functions and relations;
  - reason (from graphs, tables, and formulas) about functions derived from other functions via transformation (e.g., $g(x) = 3\,f(x - 2) + 5$), inversion, composition, and arithmetic combination.

- **Use symbolic forms to represent and analyze mathematical situations and structures**

  In grades 9–12, all students should—

  - represent situations that involve variable quantities with expressions, equations, inequalities, and systems of equations using a variety of equivalent forms;
  - develop fluency operating on polynomials, vectors, and matrices using by-hand operations for the simple cases and using technology for more complex cases;
  - understand symbolic algebra as abstracted arithmetic;
  - be able to explain, compare, and contrast the major properties of the objects and operations defined within and across systems (e.g., rational numbers, polynomials, matrices, and functions) as they follow certain rules or laws of structure;
  - develop strategies for deciding whether symbolic results generated with technological tools are reasonable, and interpret such results in meaningful ways.

- **Use mathematical models and analyze change in both real and abstract contexts**

  In grades 9–12, all students should—

  - model a wide range of phenomena with a variety of functions including linear, quadratic, exponential, rational, trigonometric, and recursively defined functions and recognize that a particular type of function can model many different situations;
  - approximate and interpret accumulation and rates of change, both graphically and numerically, for functions representing a variety of situations;

- approximate and find intercepts, local extreme values, and asymptotic behavior of functions, and interpret such results in given contexts.

Discussion

- **Patterns, functions, and relations**

Work in patterns, functions, and algebra in grades 9–12 builds substantially on students' prior experiences. Interpretation and reasoning about graphical, numerical, and symbolic forms should be a regular feature in high school classrooms. Students should be able to move flexibly among the various representations of a function. The following example requires a shift from verbal to graphical representation.

> A flight from SeaTac Airport in Washington to LAX Airport in Los Angeles has to circle LAX several times before being allowed to land. Plot a graph of distance of the plane from Washington against time from the moment of takeoff until landing.

Students with an undeveloped notion of function often lose sight of the variables they are representing in the graph and confound the graph with the path of the plane, as shown in figure 7.4.

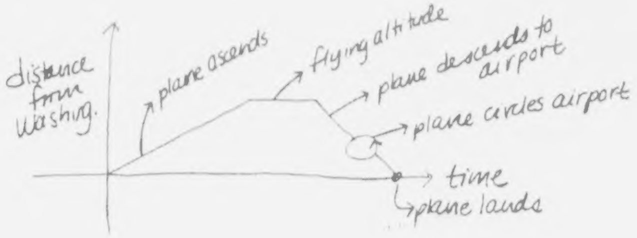


Figure 7.4. Student solution to airplane problem.

After a class discussion of various student representations of this situation, students should be helped to focus on the relationships of the quantities being represented in the graph so that they see that the circular path of the plane will not result in a circle in the graph. Rather, it will result in an oscillation in the distance from the originating point, as shown in figure 7.5. With this solution, students might then be asked to estimate the distance of the plane from the airport at a given time or to list three times at which the distance of the plane from the airport is the same.

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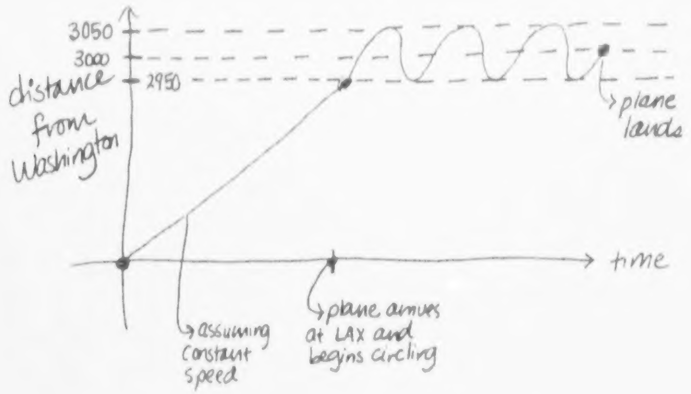


Figure 7.5. Revised solution to airplane problem.

The availability of technology allows students to ask questions involving deeper examination of the properties of families of functions. With graphing utilities, students can quickly zoom in and out on the graph of a function to view both local and global behavior. Since it is often difficult to merge both local and global views clearly, students must understand functional behavior in order to solve problems such as the one shown in figure 7.6.

> A cubic polynomial is graphed in the window $x \in [-10,10]$, $y \in [-10,10]$. The screen reveals two lines that appear to be vertical. There are exactly two roots in the diagram already. Does there need to be another? If so, where might it be? Sketch the possibilities for the complete graph.

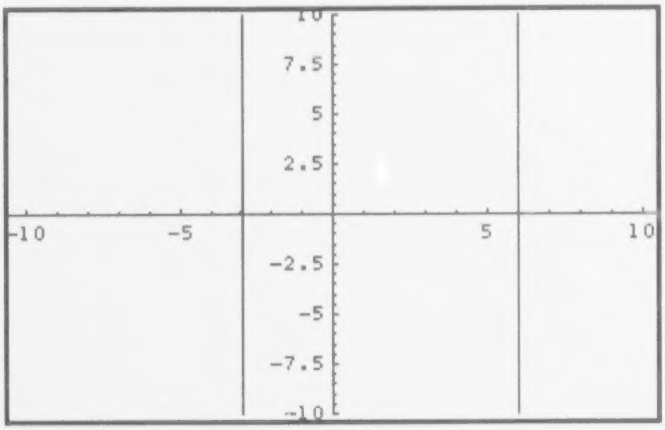


Figure 7.6. Partial graph of cubic polynomial.

Graphing software can help students examine how changing a parameter affects the graph of a function. The study of parameter changes, helps students understand ways that families of functions may be represented, as in the following problem:

- Explore the graph f(x)=nx^2 for different values of n. How do changes in n affect the graph?
- Consider how you might represent this family of functions using an "n-axis" to show how the graph changes as n changes.

Knowledge of the function family should allow the student to understand the three-dimensional graph in figure 7.7. Students might be asked to draw connections between this graph and a representation on the xy-plane of the some members of the family, given by specific values of n, thus paving the way for discussion of topographical maps and other uses of level curves to represent three-dimensional phenomena in two dimensions.

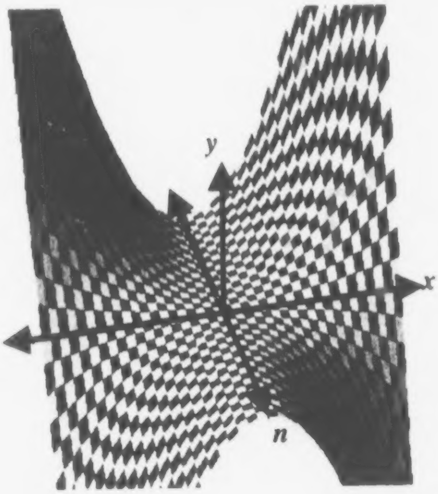


Figure 7.7. Three-dimensional representation of f(x) = nx^2

Students can use their understanding of functional concepts to solve problems that might be interesting to them. For example, a teacher might help students bring together data collection, measurement, and function ideas by setting up an experiment for discovering the effect of hill steepness on falling time. In this experiment, a flat board that is 3 meters long is set up as a ramp that can be adjusted to various heights between 0 and 100 centimeters. Students release a skateboard at the top of the ramp and time the skateboard until it stops. Ramp height should be varied throughout the experiment to heights such as 15 cm, 30 cm, 45 cm, and 60 cm. Several runs should be completed at each height and recorded in a table. Students should then be able to graph the data and make estimates for run times at other ramp heights. Teachers should encourage students to discuss what factors might explain differences between the time estimates and the actual run times, or to estimate the run time for a 90-cm ramp height. Once students have conducted the experiment and collected the data, they can use a technological curve fitting tool to generate possible function rules to describe the data and then decide which function family best describes the situation. This discussion can cause them to think about the physical situation as it relates to various function possibilities; for example, a linear model is not a good fit because the rate of change is not constant (adapted from Fey et al. 1999).

- **Structure and symbolic notation**

People who understand fundamental algebraic principles and the symbolism of algebra use it for many reasons: to generalize their calculations, to have a record of their thinking, to externalize their thinking when the process is too complicated to manage mentally, and to represent their thinking to others. Students in high school need to acquire facility with the symbolism of algebra and need to use the properties inherent in algebraic structure in a wide range of situations. This requires a combination of experience in doing paper-and-pencil manipulation of algebraic expressions and solution of equations, and experience in using technological tools, such as computer or calculator algebra systems, for the expression and solution of equations.

Facility in manipulating algebraic symbols is a useful if not necessary tool for the solution of interesting and practical problems such as the following. Suppose that a nonprofit organization has just received $40,000 to be used toward a project. The organization assesses a 19% overhead charge against all direct project costs. When a high school student is asked to decide how much money the organization actually has to spend on the project, he or she might be pointed toward a solution that relies on algebraic symbols and principles; that is, if C represents the amount that can be spent on direct costs, then $C + .19\,C = 40{,}000$. Using algebraic principles, including distributivity, to calculate C, the student finds that only $33, 613.44 can be spent on project costs.

Success in the example shown in figure 7.8 (adapted from Tucker 1995) requires more than symbol manipulation.

If $f(x) = x^2 - 1$ and $g(x) = (x + 1)^2$, complete the table below.

x	$f(x)$	$g(x)$	$f(g(x))$	$g(f(x))$
2			80	16
		4		81

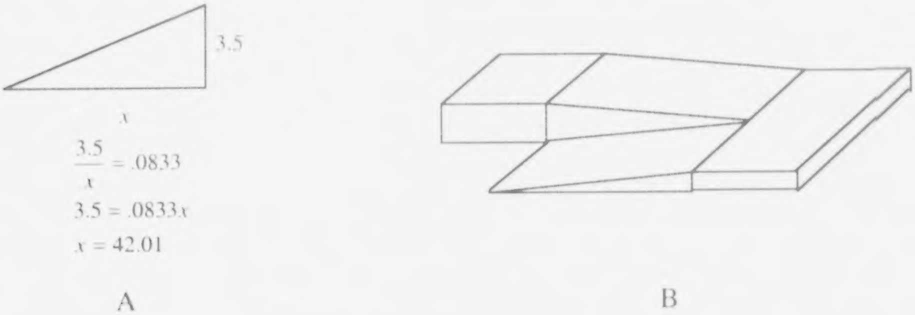
Figure 7.8. Composition of functions problem.

To complete the first row of this table, students need only to know how to compute $f(x)$ and $g(x)$ given x. However, to complete the second row, students must know what the symbols $f(g(x))$ and $g(f(x))$ represent and what it means to compose functions, as well as understand the role of the "inner" and "outer" function in a composition. Students may reason, using intuitive understanding of the idea of an inverse of a function, that since $g(x) = 4$, then x must be either 1 or –3. They can then determine that x cannot be 1, because $g(f(1))$ is 1, not 81. Alternatively, students might note that since $g(f(x))$ is 81, $f(x)$ must be 8 or –10. Since $f(x)$ is always greater than or equal to –1, it must be that $f(x) = 8$. Then x must be either 3 or –3. But since $g(3) = 16$ and $g(-3) = 4$, the only solution is $x = -3$. Thus, there are several ways to approach this problem, and each requires fluency in algebraic symbols, concepts, and reasoning.

The symbolism of algebra can be useful in recording information from real situations in a way that enables analysis conveniently. Consider what students might do with the following problem:

> The maximum gradient of ramps for handicap access is 8.33%. (That is, the slope is 0.0833.) The maximum ground distance allowed for a single ramp at the maximum gradient is 30 feet. Design a ramp to a door which has its sill 3.5 feet above the surface. What is the minimum length ramp that can be used in this situation? (Adapted from Core Plus 1997.)

The teacher might guide students by asking, "How long a ramp would be needed to rise 3.5 ft. at the maximum gradient?" Using their knowledge of slope, students might draw a diagram and write and solve equations, as shown in figure 7.9A. Then, realizing that one 30 foot ramp will not be sufficient, they might arrive at a design using two ramps, as in figure 7.9B.



$$\frac{3.5}{x} = .0833$$

$$3.5 = .0833x$$

$$x = 42.01$$

Figure 7.9. Solutions for ramp problem.

The importance of symbolic reasoning is especially evident to students in the context of applied settings. Consider the following problem:

> Each year your school has a special campaign to sell school T-shirts. Suppose that you have the task of ordering a supply of shirts and setting the price. Your goal is to arrange plans to maximize profit for the project. Estimate the demand function by surveying your class and finding out how many shirts would be purchased at each of the following prices. Then use the actual number of students in the school to estimate the number of shirts that would be sold at each price.

Price in dollars	Total number of shirts your classmates say they will buy at this price	Average number of shirts bought by each classmate	Projected number of shirts for the whole school
4			
8			
12			
16			
20			

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Students could pursue questions such as the following:

- Find a demand function $s(p)$ that predicts sales of T-shirts as a function of price p.
- Find a rule that predicts revenue as a function of price.
- Find the price that gives maximum revenue. Then calculate the number you expect to sell and the total revenue at that price.
- Assuming T-shirts cost $5 each, find a rule that gives cost as a function of price.
- Find a rule that predicts profit as a function of price.
- Find the price that gives maximum profit. Then calculate the number you expect to sell and the total profit at that price.

The answers to these questions can give rise to deeper investigation. Were the price that maximized revenue and the price that maximized profit the same? Why did it turn out that way? What if a quantity discount for the number of T-shirts is offered? Students can combine knowledge of symbolism, structure, and the context to pursue such questions. (Adapted from Heid et al. 1995.)

Students in grades 9–12 develop increasingly robust understandings of the concepts of variable, expression, equation, and function. There is additional focus on developing their symbolic reasoning capabilities and their fluency in symbolic manipulation, including the ability to understand algebra as generalized arithmetic and the ability to recognize and reason about symbolic forms.

- **Change and modeling**

The concept of a function's rate of change should be extended in high school mathematics. The example given in Figure 7.10 foreshadows ideas of calculus. Both the graphical and numerical information suggest that $y = x^2$ is nearly linear for values of x very close to 3.

Describe the graph $f(x) = x^2$ using the window $x \in [2.995, 3.005]$, $y \in [8.97, 9.03]$. Does it look like a parabola?

Consider the table of data (values of x^2 are rounded to three decimal places). Do the data support your response? Create a graph of the values in the table.

x	2.995	2.996	2.997	2.998	2.999	3.000	3.001	3.002	3.003	3.004	3.005
$f(x)=x^2$	8.970	8.976	8.982	8.988	8.994	9.000	9.006	9.012	9.018	9.024	9.030

How would you use the data to approximate the "slope" of $f(x) = x^2$ at $x = 3$?

Figure 7.10. Examining change.

Thinking about the concept of "slope" for a parabola can lead to the idea of instantaneous rate of change, a central concept in calculus. This is also a setting in which students' understanding of linear functions can be reinforced and in which connections among graphs, tables, and symbols can be highlighted.

Another important use of functions is in mathematical modeling. Modeling involves deciding on the relevant features of a real-world situation, representing those features with functions, analyzing and reasoning about the properties of the situation being modeled, and critiquing the fit of the model to the situation being modeled.

Some modeling problems give rise to iteration, an important concept for studying repetitive processes. Spreadsheets are quite useful in solving such problems, many of which may be used with students at various levels of high school, using approaches that depend on students' sophistication with symbolic reasoning and with such ideas as writing the expression for a function recursively. Consider the following example:

> A student strained her knee in an intramural volleyball game, and her doctor prescribed an anti-inflammatory drug to reduce the swelling. She is to take two 220 milligram tablets every 8 hours for 10 days. Her kidneys filter 60% of this drug from her body every 8 hours. How much of the drug is in her system after 10 days? (National Research Council 1998, p. 80).

A student might approach this problem by creating a table of values relating the number of 8-hour periods to the amount of drug remaining in the student's system (see table 7.1). This table can be continued through 10 days. Students at early stages in their study of algebraic notation might note that a value in the a_n column is found by multiplying the previous value by 0.4, and adding 440, thereby generating a need for more convenient notation such as the recursion equation, $a_{n+1} = 0.4\, a_n + 440$, which may be used to generate the table quickly in a spreadsheet. Situations such as this can give students experience capturing the relevant variables in a problem and determining the relationships among them.

n	Day	Time (hours)	a_n = drug remaining (mg)
0	1	0	440
1	1	8	.40(440)+440=616
2	1	16	.40(616)+440=686.4
3	2	24	.40(686.4)+440=714.56
•	•	•	•
•	•	•	•
•	•	•	•
n			$a_n = .40(a_{n-1}) + 440$

Table 7.1. Partial solution to drug retention problem.

Using spreadsheets or appropriate graphing calculators, students graphing the points (n, a_n) may observe that the values are approaching a horizontal asymptote, which could help illustrate the notion of equilibrium. In the upper grades, students may explore the exponential nature of the graph and obtain a closed form equation $a_n = -293(0.4)^n + 733$. Recursively defined functions are a valuable tool in solving and analyzing real situations. Because there are situations that can be modeled recursively but where a closed form expression cannot be found, recursive patterning and reasoning should be encouraged.

Through a balanced and coherent study of functions, relations, and the symbolic language of algebra, students should complete high school able to recognize equivalence among expressions and to convert flexibly among various representations of functions. They should be able to recognize the structure of a problem, represent it mathematically, and interpret the results. Teachers should capitalize on the opportunities afforded by technological tools to advance students' understanding of these concepts and to develop the mathematical capabilities needed in the world.

Standard 3: Geometry & Spatial Sense

Mathematics instructional programs should include attention to geometry and spatial sense so that all students—

- **analyze characteristics and properties of two- and three-dimensional geometric objects;**
- **select and use different representational systems, including coordinate geometry and graph theory;**
- **recognize the usefulness of transformations and symmetry in analyzing mathematical situations;**
- **use visualization and spatial reasoning to solve problems both within and outside of mathematics.**

Elaboration: Grades 9–12

Geometry teaching in grades 9–12 should build on student experiences from earlier grades to make more formal and complex levels of geometric knowledge accessible. Students' previous geometric knowledge becomes deeper, they acquire important new tools that can be used in a wide range of applications, and they become proficient in proof.

The classes of objects that form the core of high school geometry (lines, angles, polygons, circles, and a variety of three-dimensional objects) are much the same as in previous grades. Geometry learning in grades 9–12 should be focused more on the relationships among these objects than on the objects themselves or their individual properties.

A critical element in the study of geometry for students in grades 9–12 is knowing how to judge, construct, and communicate proofs. Electronic technology enables students to explore geometric relationships dynamically, offers students visual feedback and measurement as they investigate geometric situations. Secondary school teachers face an important challenge in balancing this use of technology for exploring ideas and developing conjectures with the use of deductive reasoning and counter-example in establishing or refuting the validity of such conjectures. Students should be able to articulate for themselves why particular conclusions about geometric objects or relationships among objects are logically sound. It is not critical that students master any particular format for presenting proofs (such as the two-column format). What is critical, however, is that students see the power of proof in establishing general claims (theorems) and that they are able to communicate their proofs effectively in writing.

Students should understand a range of representations, including coordinates, trigonometric relationships, networks, transformations, vectors, and matrices, which they can flexibly apply in situations that are inherently geometric and where geometry is not the original setting. Not all of the geometric situations that students encounter lend themselves to analysis within a Euclidean system. Alternative systems, such as discrete graphs, spherical geometry, or fractals can provide avenues for exploring such situations. The study of geometry in high school provides a means of describing, analyzing, viewing, and understanding the physical world and seeing beauty in its structures.

Focus Areas for Grades 9–12

- **Analyze characteristics and properties of two- and three-dimensional geometric objects**

  In grades 9–12, all students should—

  - explore relationships among, make and test conjectures about, and solve problems involving classes of two- and three-dimensional geometric objects;
  - connect geometry to other strands of mathematics (e.g., measurement, algebra, trigonometry), relate it to other areas of interest (e.g., art, architecture), and use it to solve problems;
  - recognize geometry as an example of a deductive system, built from undefined terms, axioms, definitions, and theorems; and use deduction to establish the validity of geometric conjectures and to prove theorems.

- **Select and use different representational systems, including coordinate geometry and graph theory**

  In grades 9–12, all students should—

  - investigate and verify conjectures and solve problems involving two- and three-dimensional figures, represented with rectangular coordinates;
  - explore other coordinate systems (e.g., navigational, polar, spherical) and their uses;
  - explore discrete/finite geometry systems (networks) and their characteristics and applications;
  - use trigonometric relationships to solve problems.

- **Recognize the usefulness of transformations and symmetry in analyzing mathematical situations**

  In grades 9–12, all students should—

  - represent translations, reflections, rotations, and dilations/contractions of objects in the plane using sketches, coordinates, vectors, or matrices and use these representations to gain information about the transformation;
  - extend transformations to three-dimensions, to include reflectional and rotational symmetry of solids;
  - understand transformations (under the operation of composition) as an algebraic system of functions.

- **Use visualization and spatial reasoning to solve problems both within and outside of mathematics**

  In grades 9–12, all students should—

  - draw and interpret two- and three-dimensional objects including those involving overlapping figures/objects and those requiring auxiliary lines;
  - analyze cross-sections, truncations, and compositions/decompositions of three-dimensional objects;
  - visualize three-dimensional objects and spaces from different perspectives.

Discussion

- **Shape and structure**

By high school, students should have command of a broad range of geometric objects and properties. They should have had experience making and justifying simple conjectures about the relationships among those objects. During grades 9–12 the complexity of the relationships, whether presented statically or dynamically, must be extended and deepened. Using dynamic geometry software or physical models, students can quickly explore a range of examples. They can analyze what seems to change and what seems to remain invariant, and they can create conjectures about a given geometric situation. For example, students may notice that the diagonals of a parallelogram appear to meet at their respective midpoints. Many students may be content to stop at this point, convinced that their observation must generalize because it works for so many examples. However, effective teachers must challenge such assumptions. They can take advantage of these opportunities to encourage students to develop deeper understandings by formulating verifiable conjectures, exploring possible explanations, and finally resolving them through counterexamples or proofs.

Dynamic geometry software offers opportunities for exploration and conjecture. For example, using such software students can draw a triangle and construct the midpoints of the sides. The midpoints of the three sides can then be joined and the ratio of the area of the midpoint triangle to the area of the original triangle can be computed (see figure 7.11A). As students drag one vertex to create many different triangles, the ratio appears to remain constant at 0.25. Whether this relationship always holds is a question; resolving the issue calls for logical argument and proof by the student. The proof might involve either a synthetic representation calling on properties of transversals of parallel lines or an analytic representation using algebra.

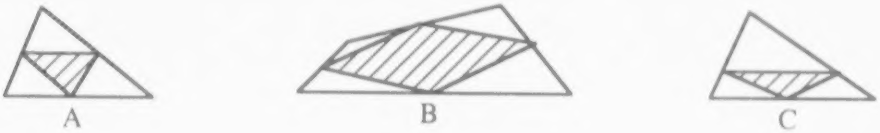
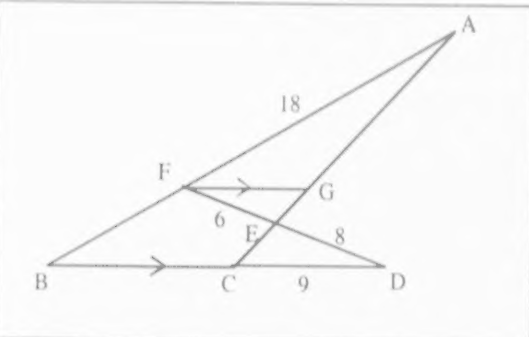


Figure 7.11. Exploring midpoint results.

One natural extension is to look at whether any such constant ratio exists for convex quadrilaterals. After testing this conjecture, it appears that the ratio of areas is also constant (this time 0.5) in the quadrilateral case (see figure 7.11B). What about a convex pentagon? Students who are encouraged to explore this idea should discover that, when the midpoints of the sides of a general pentagon are joined, no constant area ratio emerges. A counterexample should then be the expected result of the investigation (Zbiek, 1996).

A second option for extension is to use the midpoint on one side of the triangle connected to a trisection point of each of the other two sides as shown in figure 7.11C. (And constructing the trisection point is a challenging problem in itself.) As other proportional divisions of the triangle are explored, patterns of the area ratio are interesting to analyze. In this type of exploratory environment, students can pose questions for extension and further exploration for themselves.

Students in grades 9–12 should be able to answer questions and prove theorems about geometric situations even when the diagrams that depict them are somewhat complex. Teachers might modify a task such as the one in figure 7.12 to assess students' inclinations to find relationships in a more open-ended way. For example, teachers might provide only the figure and ask students to find a pair of congruent triangles or a pair of similar triangles, or to list other relationships that hold for the figure. Students can then be asked to justify their claims on the basis of earlier theorems and facts.



In the figure to the left, AF = BD, FG = FB, and FG is parallel to BD.

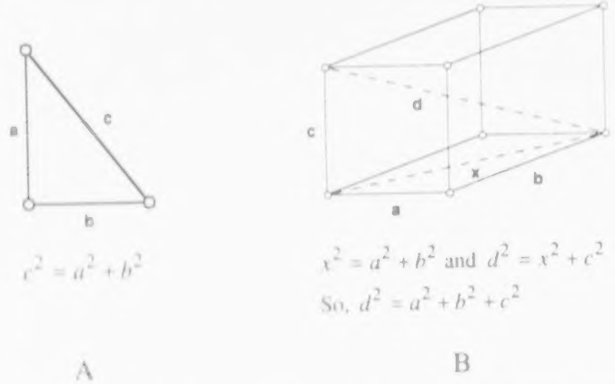
a) Name the triangle that is congruent to ΔAFG,

b) Prove that ΔAFE is similar to ΔDCE, and

c) Given that AF = 18 cm, FE = 6 cm, CD = 9 cm, and DE = 8 cm, calculate CE and EG

Figure 7.12. Complex geometric figure (from Beng 1995).

Analogies between one-, two-, and three-dimensional situations offer opportunities for expanding student understanding. For instance, students could think about how the two points which are equidistant from a fixed point on a number line (one dimensional situation) are analogous to a circle when the fixed point is located on the plane (two dimensional situation) and a sphere when the point is located in space. Using the Pythagorean Theorem to compute the distance between two points in two dimensions (see figure 7.13A) has a three-dimensional counterpart, as in figure 7.13B. Applying the Pythagorean Theorem twice yields a generalized method for computing the distance.



$$c^2 = a^2 + b^2$$

$$x^2 = a^2 + b^2 \text{ and } d^2 = x^2 + c^2$$

$$\text{So, } d^2 = a^2 + b^2 + c^2$$

Figure 7.13. The Pythagorean Theorem in two and three dimensions.

It still describes the distance between two points, but in space the points might be opposite corners of a rectangular box and would require three coordinates if described analytically.

Geometry provides methods for visualizing problems that come from non-geometric settings. For example, problems in probability can sometimes be made more accessible when viewed geometrically. The problem in figure 7.14 requires use of the triangle inequality.

A line segment, 18 inches long, is randomly divided into three lengths. What is the probability that the three lengths could be used as the sides of a triangle?

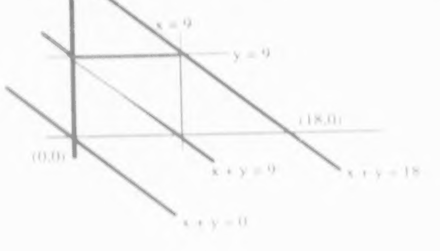


Figure 7.14. Probability of making triangles.

Suppose x and y represent two of the three lengths. Since they are lengths, they must be nonnegative. Since the total of all three sides is 18, their sum must be less than 18. Thus all possible values for x and y must be contained in the region bounded by $x > 0$, $y > 0$, and $x + y < 18$—i.e., within the triangle with vertices (0,18), (0,0), and (0,18) shown in figure 7.14. To form a triangle, the Triangle Inequality implies that both x and y must be less than 9, but their sum must be greater than 9. Thus the shaded triangle in figure 7.14 contains all possible solutions for x and y so that a triangle is formed. Since the shaded triangle has 1/4 the area of the larger triangle, there is a 25% probability that a triangle will be formed.

Applications of geometry in practical situations abound. Designs of optimal packaging, machine tools and parts, art, and architecture all rely on geometric analysis for their creation. The natural physical world around us is more understandable when viewed through a geometric lens.

- **Coordinate geometry and other geometric models**

Solving geometry problems often involves making a choice among a few possibilities for representation. Students in grades 9–12 need to develop enough facility in understanding various models so that they can benefit from alternative approaches to solving problems, alternative paths to understanding concepts, and the rich connections that such models afford.

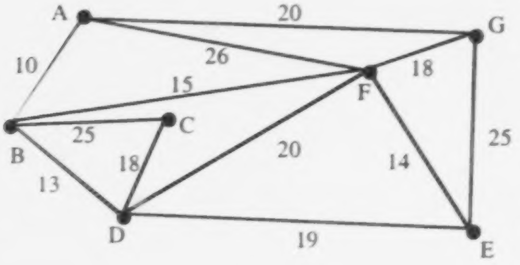
Analytic geometry offers students a powerful way of linking algebraic and geometric concepts, thus enriching both. For example, when generalizing possible roots for quadratic equations, an examination of the graph of a parabola makes it clear that only three cases exist: two roots if the parabola crosses the x-axis, one if it is tangent, and none if it does not intersect the x-axis. On the other hand, the quadratic formula gives an algebraic interpretation of these cases. The quadratic formula may be stated as follows:

$$\text{If } ax^2 + bx + c = 0 \text{ and } a \neq 0, \text{ then } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Students should see that this formula gives rise to two real roots, one real root, or two complex roots, depending on the value of the expression under the radical. Furthermore, these cases correspond exactly with the three geometric cases listed above. The expression $b^2 - 4ac$ is called the *discriminant* precisely because it discriminates among these possibilities.

High school students should also explore problems for which a polar, navigational, or spherical coordinate system might prove to be a more helpful representation. Right triangle trigonometry, including the Laws of Sines and Cosines, is useful in establishing a wide variety of area formulas. It also provides natural links to circular functions in algebra and to measurement topics and encourages yet another approach to solving problems. Students should be comfortable moving between any and all of these representational systems and making judgments about the advantages of each for a particular problem.

Two additional representation systems arise from more contemporary mathematics and should be part of the geometry that students encounter in high school. The first of these is finite graphs (structures of vertices and edges) together with their associated matrix representations. In problems where a finite number of objects or locations are to be linked by paths (networked), discrete graphs provide an effective model for answering questions about the number of possible routes, the lengths of those routes, and the optimal route under given conditions. Algorithmic thinking can be enhanced by considering methods for satisfying these conditions. An example follows.



There are seven small towns in Smith county that are connected to each other by dirt roads, as in the diagram to the left. (The diagram is not to scale.) The distances are in kilometers. The county, which has a limited budget, wants to pave some of the roads so that people can get from every town to every other town on paved roads, either directly or indirectly, but so that the total number of miles paved is minimized. Find a network of paved roads that will fulfill the county's requirements. Eliminate any non-paved roads from your drawing. (adapted from Coxford 1996)

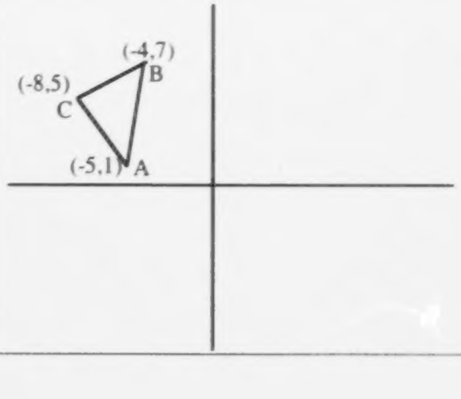
- **Transformations and symmetry**

By high school, student should have experience with basic geometric transformations: translation, reflection, rotation, and dilation/contraction. High school students should be ready to engage with increasingly general ways of representing these transformations, including graphs, notation such as R_{90°, and matrices. Matrix representation allows study of transformations as objects in their own right and illuminates natural and powerful connections to concepts found in number and algebra. Consider how the following extended example might help students understand matrices as representational tools, and also gain deeper understanding of geometric transformations. Students might be asked to read the following explanation and fill in details so that they can explain it adequately to themselves and their peers.

$\triangle ABC$ with A(-5,1), B(-4,7) and C(-8,5) can be represented by the matrix

$$\begin{pmatrix} -5 & -4 & 8 \\ 1 & 7 & 5 \end{pmatrix}$$

TO THE STUDENT: What do the rows and columns of this matrix represent?

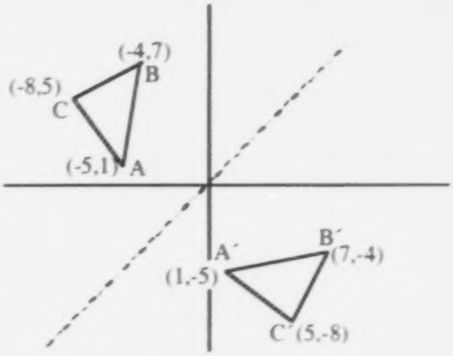


Reflection of △ABC over the line $y = x$ can be represented by the following multiplication

$$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}-5 & -4 & -8\\ 1 & 7 & 5\end{pmatrix}=\begin{pmatrix}1 & 7 & 5\\ -5 & -4 & -8\end{pmatrix}$$

so that the vertices of the image are A′(1,-5), B′(7,-4), and C′(5,-8).

TO THE STUDENT: Why does multiplication by $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ accomplish this reflection?

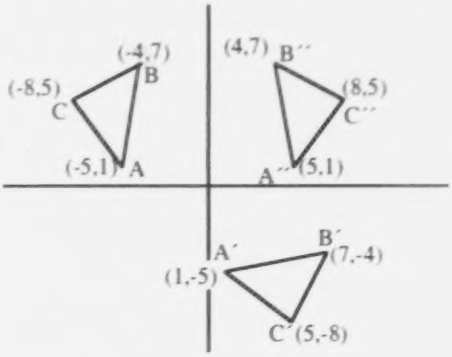


A subsequent rotation of the image 90° clockwise about the origin can be represented as follows:

$$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 7 & 5\\ -5 & -4 & -8\end{pmatrix}=\begin{pmatrix}5 & 4 & 8\\ 1 & 7 & 5\end{pmatrix}$$

so that the vertices of the second image are A′′(5,1), B′′(4,7), and C′′(8,5).

TO THE STUDENT: Why does multiplication by $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ accomplish this rotation?



Note that the product of the two transformation matrices, as shown to the right, produces the matrix of another transformation. This matrix represents a reflection over the y-axis., the transformation mapping △ABC to △A′′B′′C′′.

$$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$$

TO THE STUDENT: Use matrix multiplication to find the image of △ABC using the transformation represented by $\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$. Did the vertices of △A′′B′′C′′ result?

Matrices are a useful way to represent transformations, where their product represents a composition of transformations.

TO THE STUDENT: Explain what this means, based on the discussion above.

Note that the order of operations is important. If the two factors are reversed, as shown to the right, a different transformation will be represented. **TO THE STUDENT:** What transformation does this matrix represent? How would you compose the two given transformations to produce this transformation? Draw the image of △ABC using this composition of transformations and verify your answer using matrix multiplication.	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

There are interesting extensions for this problem. Can transformation of a quadrilateral be represented using matrix multiplication? What other transformations can be represented by matrices? Are there conditions under which matrix multiplication is commutative?

Students in the upper grades should formalize symmetry in two- and three-dimensional objects, recognizing symmetries as reflections or rotational mappings of a figure onto itself. Sets of transformations and the operations defined on them can be studied as objects in their own right. Structural questions such as closure, existence of inverses, and commutativity can be investigated and applied when analyzing sets of transformations.

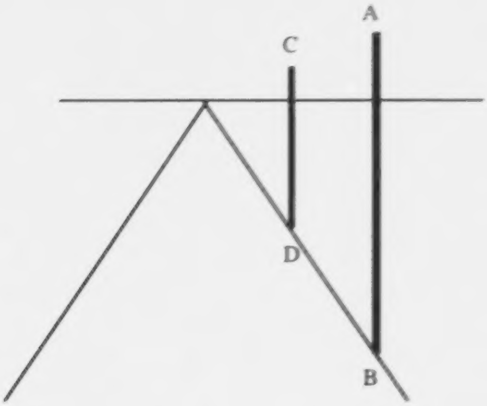
- **Visualization and spatial reasoning.**

Students must learn to pay attention to the details of a drawing or description of a physical object, and they should work to form a mental image of it if they are expected to analyze its relationship to other objects. A diagram such as the one in figure 7.12 is complex. It is difficult for a student to answer questions about it unless its key features can be identified and visually isolated. In figure 7.12, FG is parallel to BD. The line segments AB, FD, and AC all form transversals for this pair of parallel segments. Students need to be able to concentrate on the parallel segments and each of these transversals in order to identify pairs of congruent or supplementary angles and answer the questions posed. It is through a rich variety of experiences, first with simple figures and sets of directions, and later with more complex ones, that students can develop this visual ability.

While students should arrive in high school students with experience creating three-dimensional objects from two-dimensional patterns (nets), finding volume, and computing surface area, additional concrete experience is necessary in grades 9–12 to help students extend their understandings of objects in space. Visualizing such things as the solid swept out when a plane figure is rotated about an axis, the shape of a cross-section formed when a plane slices through a cone (a conic section) or other solid object, or the shape of the two solids formed by such a slicing process, requires that students work with physical models as well as drawings. Visualization software can also be helpful in this endeavor.

Interesting geometry problems set in a context of perspective drawings can build visualization skill as well as geometry knowledge. Consider the following problem (adapted from ARISE):

In the drawing below are two telephone poles drawn in perspective. We would like to draw another pole, EF, that appears to be the same distance from pole CD as pole CD is from pole AB.



If students think about the situation in two dimensions, they may recognize that ABFE will be a rectangle, and that the diagonals of that rectangle will intersect at the midpoint of CD, as shown in figure 7.15A. Applying these relationships to the situation shown in perspective may clarify what can be done, as shown in figure 7.15B.

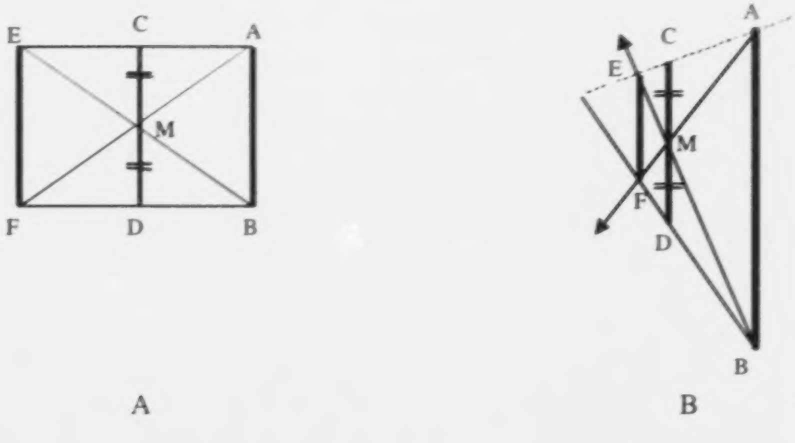


Figure 7.15. Solutions to the telephone pole problem.

Thinking about and depicting three-dimensional objects is difficult for many students. Creating and analyzing perspective drawings, thinking about how lines or angles are formed on a spherical surface, and working to understand orientation and drawings in a three-dimensional rectangular coordinate system all contribute to students' abilities to think and reason spatially. These, along with two-dimensional experiences and a solid core of deductive reasoning should be key components of the high school geometry experience.

Standard 4: Measurement

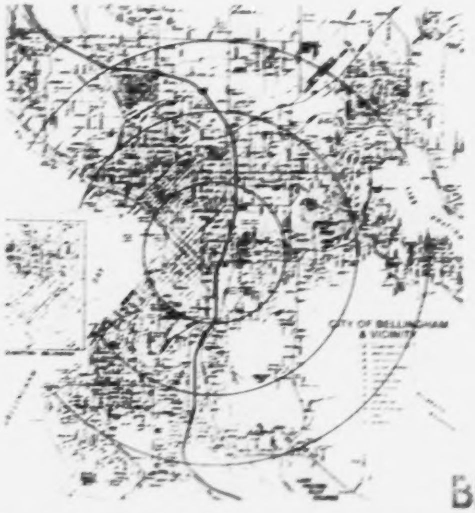
Mathematics instructional programs should include attention to measurement so that all students—

- **understand attributes, units, and systems of measurement;**
- **apply a variety of techniques, tools, and formulas for determining measurements.**

Elaboration: Grades 9–12

Typically, measurement is viewed as the process of assigning one or more numbers to the properties of an object, such as describing the weight or dimensions of some object, or describing its velocity and acceleration. In the secondary grades, the notion of measurement is broadened.

Much of mathematics consists of defining the problem space within which a problem situation exists and then choosing a representation that allows for efficient measurement of portions of this space. Most commonly used structures in grades 9–12 are the real number line, the rectangular coordinate system, and the polar coordinate system. For example, a problem involving traffic density on a portion of a freeway could be reasonably modeled with a number line using the mileage markers to determine coordinates. A problem involving travel in a city might be modeled using a rectangular coordinate system based on the city block as a unit, though measurement of distances depends upon whether the block-lengths are uniform and whether distances may be measured through the city blocks. A problem involving the population density surrounding a city could also be modeled on a rectangular coordinate system. However, the structure of a polar coordinate system would be more convenient in this case, since population density tends to decrease with the radial distance from a large city. Figure 7.16 shows the population density around the city of Bellingham.



BEST C

Figure 7.16. Population density around Bellingham.

In addition to choosing an appropriate representation, high school students should be able to choose a unit appropriate to the problem at hand. For example, the velocity of an insect measured as 4 cm/sec is easier to understand than the equivalent 0.00004 km/sec. Derived measures such as population density, pressure, or acceleration can be formed by combining various direct measurements. Particular attention should be paid to ratios of measurements, such as rate of change. Indirect measurements such as instantaneous velocity and the area inside an irregular shape may be approached via successive approximations.

Notions of scale and change of scale should be considered at the high school level. Examples of scale changes include setting the window on a graphing calculator in order to observe local or global behavior of a function, or using logarithmic scaling to straighten the graph of exponential data. Students should be aware of measurement errors and how they are compounded within calculations, and they should be able to choose an acceptable magnitude of error for a given problem situation.

Focus Areas for Grades 9–12

- **Understand attributes, units, and systems of measurement**

  In grades 9–12, all students should—

  - select an appropriate unit of measurement or scale and understand the effects of the choices that are made;
  - analyze how changes in the measurement of one attribute of an object relate to others, such as how the change in the radius or height of a cylinder affects the surface area or volume of the cylinder;
  - understand rate of change as a quotient of two different measures;
  - use successive approximations to find areas and instantaneous rates of change.

- **Apply a variety of techniques, tools, and formulas for determining measurements**

  In grades 9–12, all students should—

  - apply scaling techniques to view a problem from different perspectives, such as window changes in the graphs of functions;
  - use radian and degree measures;
  - understand and apply the concepts of variance and standard deviation as measures of spread in a distribution;
  - use dimensional analysis for unit conversion and to verify that expressions and equations make sense;
  - determine precision, accuracy, and measurement errors; identify sources (measurement or round-off errors) and magnitudes of possible errors in a measurement setting; understand how errors propagate within computations; and determine how much imprecision is reasonable for various measurements;

- use successive approximations to illustrate and use the formulas for the volume of a sphere, a general cylinder, and a cone;
- informally apply limit concepts to further develop the concepts of area and instantaneous rate of change;
- combine measurements (e.g. length, time, mass, area, volume) using ratios to produce measures such as acceleration, velocity, pressure, and density as well as dimensionless measures such as trigonometric ratios;
- combine measurements (e.g. mass, acceleration, distance) using multiplication to produce measures such as force, work, and person-hours.

Discussion—

• Attributes, units, and systems of measurement

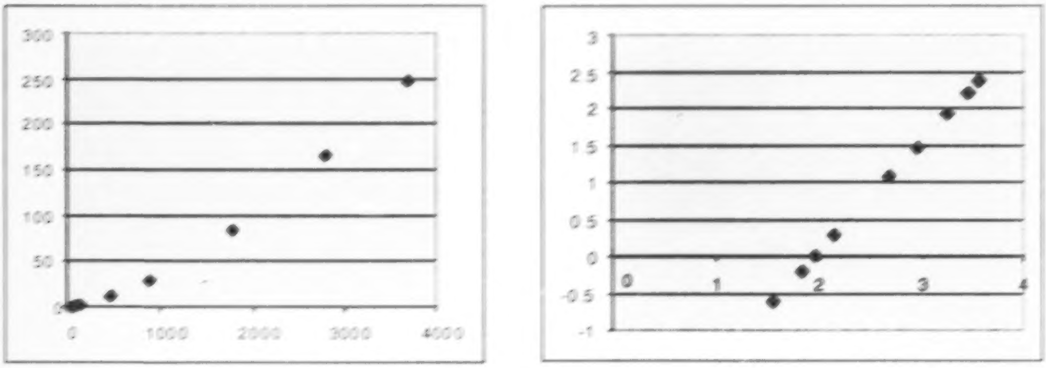
In the grades prior to high school, measurement is primarily about the assignment of a numerical value to an attribute of an object. The measurement may be determined by or understood as a comparison of the chosen unit to that attribute. This concept of measurement continues in grades 9–12 as students are introduced to radian measure and standard deviation. At the same time, the emphasis in measurement extends beyond measuring an attribute to choosing a unit or scale that allows for a clear representation of a problem. For example, logarithmic scaling can be used to represent power and exponential data in ways that "straighten" the data, thus making modeling and interpretation easier. High school students should understand when and why such scalings are used. They should also understand how composition of inverse functions plays a critical role in this straightening phenomenon.

Consider, for example, the following problem. Students are given the first three columns of table 7.2, which contains some data for the planets in our solar system. Students are then asked to form a conjecture about the number of years that an imaginary tenth planet, 10 billion miles from the sun, would take to orbit the sun.

	Length (millions of miles) of semi-major axis of orbit (x)	Time required (years) to orbit the sun (y)	log(x)	log(y)
Mercury	36	0.241	1.5563	-0.61798
Venus	67	0.615	1.82607	-0.21112
Earth	93	1	1.96848	0
Mars	142	1.88	2.15229	0.27416
Jupiter	483	11.9	2.68395	1.07555
Saturn	886	29.5	2.94743	1.46982
Uranus	1782	84	3.25091	1.92428
Neptune	2793	165	3.44607	2.21748
Pluto	3670	248	3.56467	2.39445

Table 7.2. Planet data.

When the data are graphed, as in figure 7.17A, they appear to be non-linear and possibly even exponential. The data for the four planets closest to the sun cannot be easily distinguished, making the relationship harder to detect. However, by graphing the logarithms of the both sets of values (a "log-log" transformation), as in figure 7.17B, the relationship can be more easily discerned since the logarithms have a strong linear relationship.



A

B

Figure 7.17. Planet distances graphed with different scales.

Students can then use a statistics package to find a regression line relating the logarithms, in this case $\log(y) = 1.500 \bullet \log(x) + -2.95235$. Using their knowledge of inverse functions they can determine that $y = 10^{-2.95235}x^{1.5} = 0.00111596x^{1.5}$. Thus, the tenth planet should take about 1118 years to orbit the sun. Students might contrast this model to a linear model without a logarithmic transformation. The prediction in that model is about 645 years. When comparing the predicted values of the two models to the actual values, it is clear that the regression with the log-log transform has produced very accurate results. In every case, the predicted value has an error of less than one percent. The error in the linear model is much greater, suggesting that 1118 years may be the better answer for the problem.

Many other phenomena are more clearly represented by similar logarithmic scaling. Human ears, for example, have the ability to differentiate between low sound intensities that are very close together, while not distinguishing as much between high sound intensities that are spread far apart. That is, people can easily hear the difference between sound intensities of 10^{-3} and 10^{0} pascals. We cannot hear much difference between sound intensities of 10^{7} and 10^{8} pascals. Because of this non-uniform hearing phenomenon, sound intensity measures are often displayed on a logarithmic scale, producing equally spaced decibel units.

Students can use dimensional analysis to keep track of units when converting measurements between units, as when 60 miles/hour is re-expressed as 88 feet/second. More importantly, dimensional analysis helps one keep track of which entities are being combined in which ways in more complex problems such as the following:

> Determine the work done by a fire-fighting airplane in lifting a cylindrical vat filled with water 200 feet into the air. The vat weighs 500 pounds. Its radius is 4 feet and its height is 6 feet.
>
> Students can begin with the formula for work done.
>
> $$\text{Work} \;=\; \text{force} \;\times\; \text{distance}$$
>
> They then can rewrite the formula using appropriate units.
>
> $$\text{ft} - \text{lbs} \;=\; [(\text{density of water}) \times (\text{volume of water}) + \text{weight of container}] \;\times\; \text{distance lifted}$$
>
> This then leads to appropriate choices of units.
>
> $$\text{ft - lbs} \;=\; \left[\frac{\text{lbs}}{\text{ft}^3} \times \text{ft}^3 + \text{lbs}\right] \;\times\; \text{ft}$$
>
> The final computation is then clear.
>
> $$\text{Work} \;=\; \left[62.4\frac{\text{lbs}}{\text{ft}^3} \times 16\pi \bullet 6\text{ft}^3 + 500\text{lbs}\right] \;\times\; 200\text{ft} \;=\; 3.864 \bullet 10^6\,\text{ft} - \text{lbs}$$

When doing problems that involve data from the real world, students should also be aware of issues of error propagation. If a meter stick is carefully used to measure the edges of a table to the nearest mm, yielding a measurement of 3.431m × 2.627m, students should understand that it is incorrect to report the area of the table surface as being 9.013237 square meters. Their measurements are precise to .0005m on each side, so the sides could actually have lengths in the following intervals: 3.431 ± .0005 and 2.627 ± .0005. Calculating with the extremes yields a maximum of 9.01626625 and a minimum of 9.01020825, which suggests that the calculation is dependable to only two decimal places.

- **Techniques, tools, and formulas for measurement**

Students in grades 9–12 can develop some understandings of basic formulas for surface area and volume, and they can engage in explorations that reveal how different attributes of geometric solids contribute to the volumes of those solids. Consider how students might explore the following problem:

> Given a circle of radius R, find the approximate measure of the central angle of the sector that should be removed from the circle so that the volume of the cone formed by the remaining portion is as large as possible.

Students can approach this problem empirically at first. The teacher can provide paper circles and instruct students to cut along the radius (after they have used knowledge from geometry verify that they have found the center) and then use a paper clip to estimate the location they believe would yield the greatest volume. They might fill the cone with rice and then move the clip to see if the rice level drops or overflows. For many students, the result is counter-intuitive. Often they guess the largest volume will occur when the height and radius of the cone are approximately the same. However, the volume formula, $V = (1/3)\pi r^2 h$, suggests that the radius of the cone makes a greater contribution to the volume than does the height, so the radius needs to somewhat larger than the height. Noting that $r^2 = R^2 - h^2$, students can write V as a function of h $\left[\text{i.e., } V = \frac{1}{3}\pi\left(R^2 - h^2\right)h\right]$, and use a graphing calculator to find the approximate height at which the volume is maximized for given values of R. Looking across these results should help students determine the proper angle and note that it does not depend on R, a fact that upon reflection may not be surprising since the cones of maximal volume should be similar.

As a follow-up, students can be assigned the related problem of determining the dimensions of the cylindrical can whose volume is 1 liter and whose surface area is as small as possible. By considering the hypothesis that the can with the smallest surface area is the cheapest to make, this problem can lead to discussion of why many cans in the supermarket have the dimensions they have.

The formal study of limits and related topics is not part of core material suggested here for grades 9–12. In the core, however, concepts of limit, area under a curve, and instantaneous rates of change can be approached informally through successive approximations. Students can develop good approximations for the circumference and area of a circle, for example. Areas can be cut into "thin" rectangles and instantaneous rates of change can be approximated by calculating the average rate of change between two points that are "close" together. In these ways, students can make sense of some complex notions at an intuitive level and establish the foundation for the more rigorous study of limits should they need it later on.

Standard 5: Data Analysis, Statistics, and Probability

Mathematics instructional programs should include attention to data analysis, statistics, and probability so that all students—

- **pose questions and collect, organize, and represent data to answer those questions;**
- **interpret data using methods of exploratory data analysis;**
- **develop and evaluate inferences, predictions, and arguments that are based on data;**
- **understand and apply basic notions of chance and probability.**

Elaboration: Grades 9–12

By the end of grade 8, students should have experience with posing questions, designing experiments and surveys to collect data, and representing and summarizing the data to help answer those questions. The emphasis should have been on univariate data with informal introduction to bivariate data. Students also should have been informally introduced to concepts of randomness and chance. As in other areas of the curriculum, these experiences will be deepened and broadened in high school. Taking advantage of their knowledge of number and algebra, students should formalize their understanding of the relationships among the graph of a distribution, and measures of shape, center, and spread. Bivariate distributions should be studied at some length, exploring the joint variation of two characteristics of the subjects in a population. Various techniques should be introduced, including finding regression curves and using technology to fit mathematical models. In general, handheld calculators and computers can be used effectively throughout students' study of data analysis, statistics, and probability to illustrate concepts, perform calculations, create representations, and provide data from simulations.

In high school, students should study the meaning and uses of probability distributions. This study will include an introduction to special distributions including the binomial and normal distributions. They should use distributions to predict the likelihood of events and also to assess the quality of an estimate based upon a sample.

Students also should learn more formally about what is involved in designing surveys and experiments to answer well-posed questions. Sample surveys select random samples from a well-defined population for the purposes of estimating a population parameter, such as a mean or a proportion. Experiments randomly assign treatments to pre-selected experimental units for the purpose of detecting treatment differences, if they exist. Randomness is used in both cases to reduce bias and to make the study repeatable so that a predictable measure of error can be calculated. It is this repeatability that allows simulation to be used to empirically study these sampling or experimental errors. This is the heart of statistics, and high school students can develop understanding of these basic ideas.

Students should revisit the issue of experimental design and look more carefully for other characteristics that can influence how well a sample can be said to represent the population from which it is drawn. Students should better understand the influence of sample size on the likely closeness of the estimate of a population statistic to its actual value. They should apply the Central Limit Theorem to better understand that larger samples narrow the range of reasonable estimates of population statistics from samples. These are examples of ways in which students apply statistical knowledge as they draw inferences about a population. An understanding of informal probability, with its ability to describe randomness, is needed since it forms the conceptual base on which statistical inference rests.

Focus Areas for Grades 9–12

- **Pose questions and collect, organize, and represent data to answer those questions**

In grades 9–12, all students should—

- design and carry out appropriate methods for gathering univariate data, both to study the distribution of a variable in one population and to compare the distributions of the same variable in two different populations;
- design appropriate methods for collecting, recording, and organizing data to obtain bivariate data in order to study the association between two variables;
- select appropriate graphical representations and numerical summaries of data;
- understand how a change in a representation (e.g., scales on a scatterplot, categories in a two-way table, and bin size of a histogram) affects the information it conveys;
- use calculators and computer applications (e.g., spreadsheets, simulation software, and statistical software) appropriately to assist in data collection, organization, and representation.

- **Interpret data using methods of exploratory data analysis**

  In grades 9–12, all students should—

  - compute, identify and interpret measures of center and spread (e.g., range, variance and standard deviation, and interquartile range);
  - describe shapes of one- and two-dimensional data sets;
  - look for symmetry and skewness, clusters and gaps, and possible outliers in data and consider their effects on the interpretation of the data;
  - recognize how sample size or transformations of data affect shape, center, and spread;
  - use a variety of representations of data, including scatterplots, frequency distributions, and two-way tables;
  - be able to recognize trends in bivariate data, visually and numerically, and use technology to determine how well different models (e.g., linear, exponential, and quadratic) fit data, while understanding that a perfect fit is unlikely for empirical data.

- **Develop and evaluate inferences, predictions, and arguments that are based on data**

  In grades 9–12, all students should—

  - understand the elements involved in finding good models for phenomena;
  - apply well-fitting models to predict unobserved outcomes;
  - evaluate conclusions based on data;
  - use data from samples to estimate population statistics;
  - use and interpret the normal and binomial distributions appropriately.

- **Understand and apply basic notions of chance and probability**

  In grades 9–12, all students should—

  - understand and compute probabilities of independent, disjoint, and conditional events;
  - understand that some phenomena are random and apply the law of large numbers to predict long term behavior;
  - use probability distributions to compute probabilities of events.

Discussion

- **Collect, organize, and represent data**

  As one group of statisticians put it,

  > Answers to specific questions of a quantitative nature do not come about easily! Students should have some idea of how difficult it is to answer questions like 'What percentage of voters think the school year should be increased in length?' or 'Does a high cholesterol diet increase the chance of a heart attack?' . . .they should be aware of the key features of a well-

> designed study in order to have some basis for making decisions on the many issues that revolve round such studies. This understanding can begin the upper elementary grades and be deepened and broadened throughout the middle and high school years. (Scheaffer, Watkins, and Landwehr 1996)

In high school, the major extension of the study of data, statistics, and probability from the earlier grades is an increased emphasis on multivariate data. Even when a study concerns only one variable (such as income) of a population, students should be aware that this variable is likely to be affected by many background variables, such as age, job experience, education, or work record. Associations between variables is one of the key elements of any statistical investigation.

The study of the joint variation of two characteristics should be a prominent part of data analysis in grades 9–12. Reflecting recent trends in statistical analysis and taking advantage of rapidly evolving technology, students learn to interpret information from two-way tables and various graphical representations of bivariate data. The ease of creating scatterplots, regression curves, and trend lines with technological tools supports their study in schools to help students see how pairs of characteristics are related. Students should analyze and represent data in ways that increase their understanding and facilitate communication.

By secondary school, the collection of data is a major focus within the study of data analysis and statistics. Students will have had experience with data provided from other sources; e.g., generated by their classmates or from experiments. In secondary school students can become more sophisticated in formulating questions and planning appropriate means for collecting data—through observations, experiments, and other means—that will be useful in answering those questions. They can learn about the importance of assumptions in sampling and the design of surveys and experiments.

- **Describe, analyze, and interpret data**

Students in grades 9–12 build on earlier experiences with various ways of displaying univariate data, so that their knowledge can become more formal and can be used to make comparisons. Students should use measures of center and spread to compare different populations, in order to see whether the studied variable has the same distribution in both. They should know why adding a constant to all observed values changes the measures of the center by that constant but does not change measures of spread. They also should understand that multiplying each observed value by the same constant yields a new standard deviation that is the product of the original standard deviation by the same constant. Students should have experience constructing standard scores from the mean and standard deviation of observed data, and then apply their knowledge of the normal distribution to understand the likelihood of an observation that is, for example, two standard deviations above the mean.

In these grades, there should be increased attention to bivariate data. The idea of shape from the univariate context now becomes direction. For example, is the cloud of points in a scatterplot tilted upward or downward? The idea of linearity can be addressed, as in the question, "Does a line through the center of the cloud look straight or curved?" And, the notion of spread becomes more sophisticated; for example, how tightly are the points spread about the center line? Similar trends can be explored in two-way tables.

Analysis of data provides a natural link between mathematics and other disciplines. For example, an examination of the U.S. population each decade, noting exponential trends, leads nicely to the question of what causes the deviations from expected trends. In Figure 7.18, the population for each decade from 1790 through 1980 is plotted and fitted to an exponential curve based on the first three points. It is interesting to speculate on why the data fall increasingly below the curve beginning around 1860. This exercise may be repeated starting around 1860 and again later, thus prompting discussions of possible historical causes.

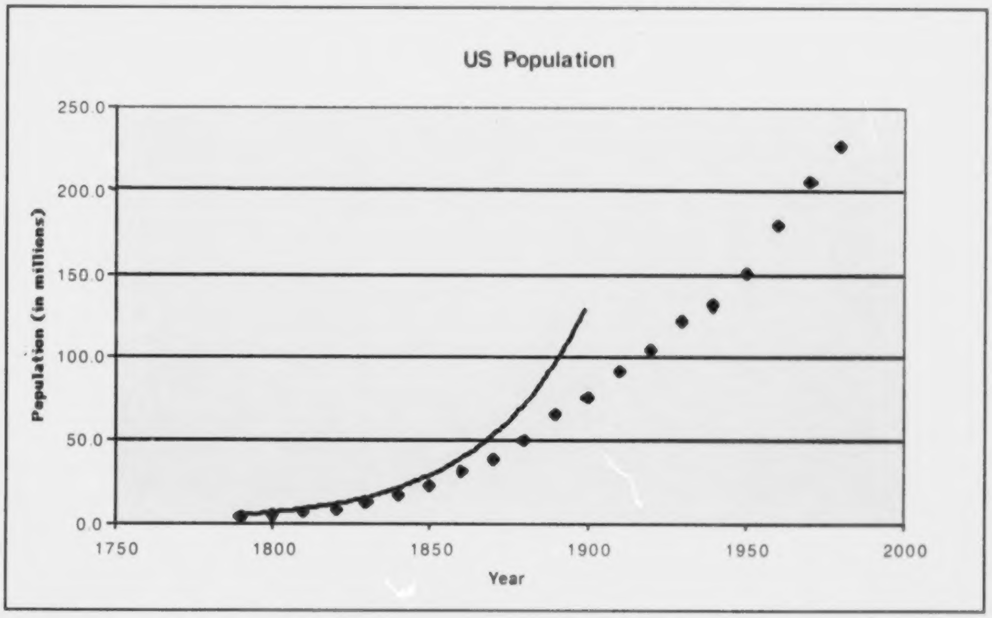


Figure 7.18. U.S. population growth.

- **Inference and prediction**

High school studies should provide students with a conceptual foundation for statistical inference, with special attention to the idea of randomness. This is a difficult concept for students to grasp. There is no natural expectation that the pattern of occurrences of an uncertain event might be predictable based on what can be learned from a sample. Statistics imposes randomness in the design of a study to reduce bias and to provide a predictable error that forms the basis for inference. Simulations that show sampling distributions for statistics, like the mean and median, are valuable instructional tools in these grades because students can see that randomness does lead to predictability in the shape, center, and spread of the sampling distribution.

As students work more with bivariate data in grades 9–12, they will be able to investigate relationships using linear, exponential, power, logarithmic, and other functions for curve fitting. Both visual estimation and correlation coefficients can help determine the best-fitting model for a data set. Students also should develop facility at recognizing outliers and appreciating their effect on regression curves and analyzing residuals. In addition to providing more mathematical tools available for making predictions, high school mathematics programs will introduce students to more sophisticated tools for comparing pairs of attributes. Correlation is an important measure of the association between measurement (continuous) data.

High school students should be prepared to carefully and critically evaluate the overall design, data collection process, and conclusions drawn from experiments, surveys, and studies. They should assess whether the words, data, and graphics contained in presentations, reports, or news articles are accurate and consistent with one another and support the conclusions presented. Students should become informed, analytical, thoughtful consumers of the information and data generated by others and should be able to collect their own data to answer important questions.

- **Probability**

Probability is counter-intuitive for most people; probabilistic reasoning needs to be addressed explicitly in order for students to become proficient at it. They need to test their intuitions against results from simulations and empirical data gathering. They need to develop the disposition to think deeply rather than trust their first impulses when confronted with probabilistic situations.

In grades 9–12, students should build upon their previous informal work with chance. Their experiments with finding probabilities by counting and computing ratios help them understand the notion of a sample space, and the notion of events as collections of points in a sample space. High school students can begin to develop an understanding of the ways of computing probabilities for combinations of events, including independent events and conditional probabilities. The following example illustrates the use of two-way tables (Hopfensperger et al. 1998).

> ELISA is a popular test for screening blood samples for the presence of HIV. For blood samples known to contain HIV, ELISA shows a positive result 99% of the time. This means that 99% of the time the test will correctly identify blood that contains HIV. For blood samples known to be free of HIV, ELISA reports a positive result about 2% of the time. This means that 2% of the time the test will incorrectly report that a blood sample contained HIV. Assume that a medical lab tests 1000 blood samples using ELISA. Also assume that 50% of the samples contain HIV.
>
> How many blood samples known to contain HIV would you expect to test positive? How many false positives would you expect to see? Copy and complete the following table:

	Contain HIV	No HIV	Totals
Tested HIV Positive			
Tested HIV Negative			
Totals	500	500	1000

High school students should learn about probability distributions. For example, they should have some experience with the binomial distribution, possibly relating it to Pascal's triangle in the area of number and to the binomial theorem in algebra. The question, "How likely is it that at most 10 of the 50 people receiving a promotion are women when the applicant pool from which the promotions are made is 65% female?" may be investigated first through simulation to obtain an approximate answer and, later, using the binomial distribution to find the theoretical solution.

The study of probability in early grades—such as with spinners—provides a strong foundation for high school students' work with density functions. The probability of an occurrence within an interval is represented by the area under the density function above the interval. This is particularly helpful for understanding the normal distribution and other continuous distributions. Probability also can be an area for connecting knowledge from other parts of the curriculum, as illustrated earlier in this chapter in Figure 7.14.

Standard 6: Problem Solving

Mathematics instructional programs should focus on solving problems as part of understanding mathematics so that all students—

- **build new mathematical knowledge through their work with problems;**
- **develop a disposition to formulate, represent, abstract, and generalize in situations within and outside mathematics;**
- **apply a wide variety of strategies to solve problems and adapt the strategies to new situations;**
- **monitor and reflect on their mathematical thinking in solving problems.**

Elaboration: Grades 9–12

The experiences that students have with problem solving in secondary mathematics education are critical to their mathematical development. No matter what body of knowledge students have, that knowledge alone will be insufficient to help them face new challenges in work, school, and life. They will have to adapt and extend what they know, and doing so effectively lies at the heart of problem solving.

Successful students have a problem-solving disposition, including the confidence and willingness to take on new and difficult tasks. Their disposition also includes a tendency to look for structure, to analyze situations carefully and represent them mathematically, and to use mathematical tools appropriately. Successful problem solvers are resourceful, both in seeking out information to help solve problems and in making effective use of what they know. Their strategic knowledge gives them options; if the first approach to a problem fails to work, they have access to a second or a third in their tool kit. If those approaches don't work, successful problem solvers know how to reconsider the problem, break it down, and look at it from different perspectives that may help them understand it better or make progress toward its solution. Part of being a good problem solver is being a good planner, but good problem solvers do not adhere blindly to plans. Instead, they monitor how well things are going and consider and make adjustments when progress does not seem to be as good as it should be. Problem solving must be a major element of the high school experience.

What is distinctive about problem solving in grades 9–12?

In grades 9–12, there are some very strong continuities with and significant extensions of students' problem solving experiences from grades pre-K–8. The general nature of students' engagement with mathematics will remain the same: they will learn about problem solving, extend their mathematical knowledge via problem solving, and solve problems.

A problem-solving approach to curriculum allows students to build new mathematical understandings through work with problems. For example, as students work on determining how many ways a network of roads between two cities can be traversed, fundamental concepts of graph theory can be addressed. Students can be asked to justify their choice of method giving rise to very rich classroom conversations in which the various methods of solving the problem are compared.

Another problem that may promote interesting mathematics is that of building a tunnel through a mountain or under a body of water, such as the English channel (the "Chunnel"). If, for example, construction teams start digging from both sides of the tunnel, how do they guarantee that they meet in the middle? There are many such problems that provide a context in which students can develop and extend their understandings of fundamental mathematics. A teacher might ask students to find a path through a city starting at one point, ending at another, and passing over each street precisely once. This problem can introduce a unit on graph theory. Similarly, notions of game theory can emerge when students play a game, determine if the game is fair, and if possible, find a strategy that determines a win or a tie.

A supportive climate for investigation helps students develop productive mathematical dispositions, and rich problems allow for deep engagement and for mathematics learning. Students' capacity for solving problems expands dramatically during high school. The set of mathematical tools at their disposal is larger. High school students learn to work with several new classes of functions. They have new processes, such as optimization and randomization, at their disposal. They learn to use new mathematical objects, such as matrices and networks. They learn to express themselves and reason with new tools, such as recursive and algorithmic thinking.

By the time they enter grade 9, students also have ready access to a number of problem-solving strategies and approaches, as well as some idea of situations in which these strategies and approaches are likely to prove fruitful. This repertoire of strategies should be continually expanding. Some strategies grow in complexity and the subtlety with which they are used. Consider, for example, a strategy which says, "look for a pattern by trying values of $n = 1,2,3,4,5\ldots$." Students can use this simple strategy in the elementary grades. In grades 9–12, the use of this strategy extends to more difficult mathematics and may be used in combination with other strategies or subject matter knowledge. A simple extension of the strategy is applicable to solve a problem such as the following:

What is the sum of the following numbers?

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\ldots+\frac{1}{n(n+1)}$$

Students might be encouraged to create a table and compute the values of the sum for small values of *n*:

Number of terms (*n*)	Sum of terms
1	1/2
2	1/2 + 1/6 = 2/3
3	1/2 + 1/6 + 1/12 = 3/4
4	1/2 + 1/6 + 1/12 + 1/20 = 4/5

Examining the data, students might conjecture that the sum of the first *n* terms is $\frac{n}{n+1}$. However, they should recognize the importance of proving their result and the power of mathematical induction in verifying statements of this sort.

Other strategies will be new to the students and they should be learned as students explore new mathematical territory. Students who are learning about cryptography, for example, can learn various strategies for coding and decoding. Some of the latter include general strategies such as decomposing and recombining or establishing subgoals. Methods such as proof by contradiction, which is often a strategy chosen for proving that something is unique, and without-loss-of-generality arguments are sophisticated and come late in students' experience, yet continue to enhance their mathematical tool kit.

What is the role of the teacher in supporting problem solving at grades 9–12?

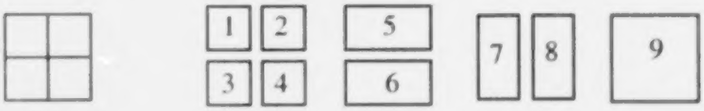
In problem solving, the main responsibility of the teacher is to establish a climate conducive to the development of students' productive mathematical dispositions. The pathway to the solution of a problem is not always apparent at the outset. Students need to be willing to dive into problems and to "get their hands dirty" in exploring aspects of the problem before they can propose potentially productive approaches. This is likely to feel very risky to students.

Students who have come from the pre-K–8 background proposed by these standards will be more likely to accept, rather than avoid, the insecurity that accompanies the tackling of complex problems. Nevertheless, students will arrive at grade 9 with great diversity in their willingness to accept insecurities and risks. A classroom environment where exploration is encouraged and where perseverance is rewarded is likely to help students become good problem solvers. With encouragement and specific feedback from teachers, students can learn to monitor themselves as they struggle with complex problems. The teacher can model questioning and promote suggestions from students, while making it clear that ultimately problem solutions must meet certain mathematical standards.

Within any curriculum, problems can be selected or modified so that they offer high potential for student engagement in mathematical thinking. It is useful to have students work on problems that can be solved in many ways. This allows for comparison of methods, identification of connections, and discussion of planning, monitoring, and adjusting choices during work on the problem. Such problems exist in every mathematical area, consider the following counting problem.

> How many rectangles are there on an 8 x 8 chess board? Be careful to count them all—any rectangle with its sides as grid lines in the chess board "counts."

One student began by working on the 2x2 chessboard:



She found that there were 9 rectangles. Students might find it helpful to note that when counting the number of rectangles on a 3 x 3 chess board they can use the their result from the 2 x 2 case, an iterative solution.

Through problem solving, students can understand the nature of mathematics. The problem in figure 7.19 illustrates the fact that different mathematical definitions have different consequences, and it raises the issue of how and why definitions are made the ways they are.

> Below you will find a collection of rectangles. Define a mathematical measure that allows you to tell which rectangle is the "most square" and which is the "least square." Define a different measure that achieves the same result. Is one measure "mathematically superior" to the other? Explain why, and be prepared to defend your choice to the class.

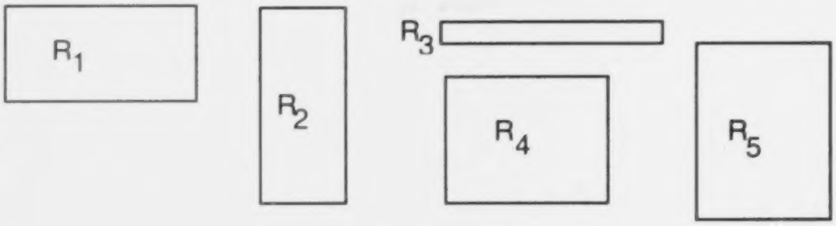


Figure 7.19 "Squareness" of rectangles.

In classrooms where open-ended problems are the norm students may dive right in. It is possible, however, that students will object, saying that none of these rectangles is "square." Encouraging students to think about what it might mean to be "almost square", having them listen to and reflect on the ideas of others, and asking them to evaluate a variety of proposals can help open students' understanding to the importance of definition in mathematical communication.

Tasks also can be framed in ways that make them more like problems than exercises. For example, any task that starts with "show that ..." can be re-framed as "A friend of mine claims that ... Is she right?" If false claims are sprinkled into the mix, the students will need to check the truth of statements rather than merely confirming them, and this is a much more mathematical enterprise.

Research and the experience of teachers indicates that if students are taught problem solving strategies and use them only in isolation, they are unlikely to use them when confronting new problems out of context (Lester 1994). Problem-solving activity must be embedded in the fabric of students' mathematics learning. Equally important is the finding that if strategies are not taught explicitly, students are not likely to learn them. Problem-solving strategies are complex and subtle, and students cannot be expected to pick them up by osmosis (Schoenfeld 1985). At various times, strategies should be the focus of instruction. They should be made explicit and the ways in which they are used should be discussed. Throughout the curriculum, students should encounter problems whose solutions depend on wise selection and good use of strategies. Teachers should discuss how to use the strategy, and also why it might have seemed reasonable to use a particular strategy on a particular problem.

The nature of classroom discourse will have a major impact on what students take away from their problem-solving experiences. The teacher can shape an atmosphere conducive to developing productive problem-solving dispositions and knowledge, both through the choice of problems and through the ways the problems are discussed. Teachers can highlight problem-solving behaviors by modeling the solution process—sometimes solving and discussing new problems from scratch at the overhead or on the board. Teachers also can highlight the importance of decision-making during whole-class discussions of problems. If a problem has been posed and a number of suggestions have been made about how to approach it, the teacher can suggest that the class take stock, consider the various options, and make a reasoned decision about which path to pursue.

Most importantly, teachers can make sure that classroom time is spent engaging in good mathematical practices. Throughout the class, teachers can pose questions that reinforce productive mathematical behaviors. For example, students tend to believe that they are done with a problem as soon as they have obtained a correct solution. Teachers can work to undo this impression by habitually asking the question, "Does anyone have another approach to this problem?" and making sure to leave enough time for students to generate alternative approaches. Teachers also could solve a problem through alternative approaches and then look for connections among them. When a range of approaches has been exhausted, the teacher can then ask, "Are there other things we can do with this problem? Can we generalize it? Can we find applications or connections? Are there related problems we might pose or explore?" Classrooms in which there is a productive problem-solving climate are the classrooms most likely to produce students who will be effective problem solvers.

Standard 7: Reasoning and Proof

Mathematics instructional programs should focus on learning to reason and construct proofs as part of understanding mathematics so that all students—

- **recognize reasoning and proof as essential and powerful parts of mathematics;**
- **make and investigate mathematical conjectures;**
- **develop and evaluate mathematical arguments and proofs;**
- **select and use various types of reasoning and methods of proof as appropriate.**

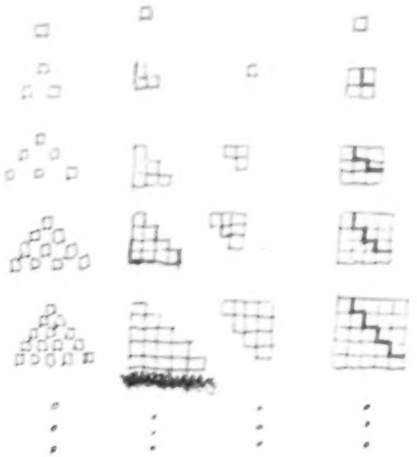
Elaboration: Grades 9–12

What is distinctive about reasoning and proof in Grades 9 to 12?

Students should emerge from their mathematics programs in grades 9–12 with a propensity to seek mathematical justifications and an ability to provide them for the range of mathematical generalizations and arguments they will encounter. Students should understand what constitutes a convincing mathematical argument. In earlier grades, students generated and tested conjectures, and they constructed and defended arguments to support those conjectures. This process should have occurred in classrooms where providing explanation and justification were a natural part of the classroom culture. As students progress through high school, they should develop an increasingly sophisticated understanding of mathematical proof and a broader array of techniques. Their standards for accepting explanations and their ability to reason in trying to understand new mathematical ideas will be stronger.

In addition to understanding that mathematical statements should be justified and should stand up to the scrutiny of peers, students in grades 9–12 should develop facility with a range of justification techniques. For example, they should continue to have experiences making conjectures and evaluating the truth of those conjectures. At the high school level, students also should be able to construct chains of reasoning to support mathematical conclusions; they should be able to step back, reflect on their arguments, and explain their ideas to others.

As students develop justifications, they should use an increasingly sophisticated range of proof-related mathematical ideas, including indirect arguments and counterexamples. Students should reason within specific mathematical areas by using ways of thinking particular to those areas (e.g., spatial reasoning, probabilistic reasoning, statistical reasoning, and symbolic reasoning). Formal proofs occur in all areas of mathematics and students' school experience with proof should not be limited to geometry. For example, a ninth grade class was exploring triangular numbers (1, 3, 6, 10, . . .) and someone noted that it seemed that the sum of any two consecutive triangular numbers would be a square. Another student supplied the following "proof without words":



Students should be able to reason about sophisticated situations, such as the sequence of digits in the decimal expansion of any rational number. This task might begin with exploring decimal representations of fractions and looking for patterns and characteristics of the length of the repeating patterns. Students should be able to explain the following result convincingly: if a number can be expressed as a fraction with denominator n, then the decimal representation eventually has a repeating pattern of length at most $n - 1$. They might use this result to observe that some specific decimal numbers are not rational (e.g., 0.0100100010000 1…), because there is never a repeating block.

How does technology affect the role of reasoning and proof in grades 9–12?

Technologically equipped classrooms provide their own particular supports and challenges for the development of mathematical reasoning. On one hand, technology gives students control in selecting examples and in producing many instances of something, thus providing the possibility of ownership of the examples and the underlying ideas. As students can check many examples to discover patterns, they need to develop ways of determining if the patterns they find are general. Although technology may help students generate conjectures, it also may make it difficult for students to see the need for a more formal justification or proof.

As students gain more experience with computer algebra systems, spreadsheets, statistical packages, and dynamic geometry tools, they also will face more occasions where their ideas need formal justification. They will need to understand the nature of approximation in computer results and to make judgments about when round-off suggests false counterexamples. They will need to have some sense of characteristics of the algorithms that underpin the generation of computer-generated results. Throughout, they need to develop an appreciation for the necessity and power of mathematical proof for establishing the truth of their conjectures.

What is the role of the teacher in developing reasoning and proof in grades 9–12?

Mathematics teachers in grades 9–12 play a critical role in helping students to develop habits of thinking and reasoning. Teachers need to understand mathematics well, and they should communicate a desire to know the reasons for mathematical patterns and truths. Their students need to see that the internal logic of a compelling mathematical argument is what establishes its correctness—not the authority of a teacher, the class, or a mathematician. For students to evaluate the effectiveness of proposed explanations, they must develop confidence in their own reasoning abilities along with the disposition to always question their own as well as others' reasoning.

As in other grades, mathematics teachers in high school strive to create a climate of questioning in their classes. Building on their students' increasing knowledge of mathematics and familiarity with conventional representations, teachers should expect their students to seek, provide, and criticize explanations so that classes become communities of inquiry. With guidance, students should develop high standards for accepting explanations. Encouraged by interaction in their classes, students should feel that they have license to develop their own arguments, instead of feeling that they must reproduce some conventional argument.

Students in effective classrooms described here can formulate their own statements and develop their own proofs. The following example describes a student who proves a special case of a general theorem that has been discussed in class. In that student's tenth grade mathematics class, all students had previously constructed curves of constant width using regular *n*-gons (where *n* is odd). They investigated several curves (including a circle and a Reuleaux triangle) with the same constant width and discovered that their perimeters had equal length. Their teacher then told them that their observations generalized to Barbier's Theorem that all curves of a specific constant width have equal perimeters. One tenth grader then proved a special case of his own, that all curves of the same constant width generated from regular *n*-gons (where *n* is odd) have equal perimeters. The student's independent work, shown in figure 7.20, was encouraged and supported by his teacher.

This example shows student-generated reasoning, in which the student takes advantage of prior knowledge and answers a question that he asked himself rather than one provided by the teacher or the textbook. The student demonstrates a disposition to question and seek understanding, a primary goal of all mathematics instruction.

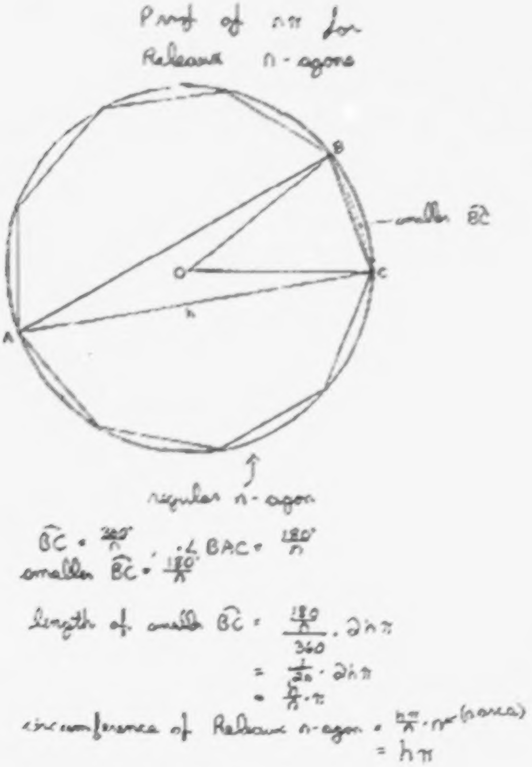

Proof of hπ for Releaux n-agons
B
smaller $\widehat{BC}$
O
C
h
A
regular n-agon
$\widehat{BC} = \frac{360°}{n}$ $\therefore \angle BAC = \frac{180°}{n}$
smaller $\widehat{BC} = \frac{180°}{n}$
length of smaller $\widehat{BC} = \frac{\frac{180}{n}}{360} \cdot 2h\pi$
$= \frac{1}{2n} \cdot 2h\pi$
$= \frac{h}{n} \cdot \pi$
circumference of Releaux n-agon $= \frac{h\pi}{n} \cdot n$ (n arcs)
$= h\pi$

Figure 7.20. Curve of constant width.

Standard 8: Communication

Mathematics instructional programs should use communication to foster understanding of mathematics so that all students—

- **organize and consolidate their mathematical thinking to communicate with others;**
- **express mathematical ideas coherently and clearly to peers, teachers, and others;**
- **extend their mathematical knowledge by considering the thinking and strategies of others;**
- **use the language of mathematics as a precise means of mathematical expression.**

Elaboration: Grades 9–12

Throughout preK–12, classroom environments that support productive mathematical discourse and that focus on clear written and oral communication help students to deepen their appreciation and their understandings of mathematics. In classrooms that foster mathematical inquiry, students develop the productive habits of systematic exploration, abstraction, and generalization. Classrooms in which students compare and contrast ideas promote the

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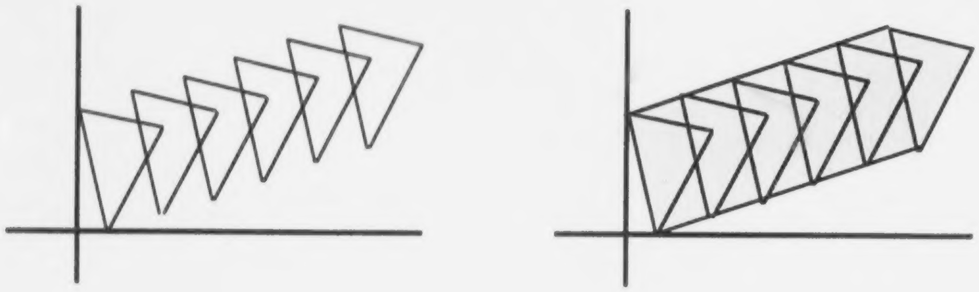
development of more careful and reflective thinkers; here, students are exposed to multiple perspectives and learn from others. Classrooms in which students are consistently encouraged and helped to think through ideas and to express them clearly produce deeper thinkers who are also more effective at conveying information to others.

What is distinctive about communication in grades 9–12?

Distinctive aspects of communication in grades 9–12 arise more from the relative maturity of the students than from changes in the nature of communication itself. In all areas, high school students should be growing in their ability to structure logical chains of thought, express themselves coherently and clearly, listen to the ideas of others, and to think about their audience when they write or speak. Thus, the major differences at this grade band are reflected in the complexity of the subject matter in which students are capable of engaging and the standards of oral and written exposition that students are expected to meet. They should be good critics and good self-critics. They should be able to generate explanations, formulate questions, and write down arguments that teachers and mathematicians would consider logically correct and coherent. Increasingly, students' oral and written language should make use of mathematical representations, terminology, and conventions. Whether students are making their point using spreadsheets, geometric diagrams, combinatorial arguments, or algebraic symbolism, they should use mathematical language and symbolism correctly and appropriately. Students also should be good collaborators who are effective at working with others.

Communication is a fundamental element of learning and is thus inseparable from any of the other standards. Students communicate about mathematical content, and such communication is a vehicle for their learning. Communication is grounded in and fosters reasoning. It is a way of making connections. It fosters problem solving and depends on the use of mathematical representations. It is, in sum, an integral part of doing and learning mathematics.

In a twelfth grade classroom, students were asked to find the area of the figure created when a triangle with vertices A(2,0), B(0,7), C(5,6) is translated by a vector $(t, 3t)$ where $0 \leq t \leq 5$. One student drew the graph shown in figure 7.21(a). Another student reacted, "No, you have to connect those, t can be one-half, too." "Oh, yeah, I forgot," said the first. Once they had drawn the figure, as shown in figure 7.21(b), the conversation turned to what shape it was: a triangle with a rectangle attached, or a triangle with a parallelogram attached.



(a) (b)

Figure 7.21. Translation of a triangle.

"How can you decide?" asked the teacher. "We could measure the angles." "No, we should check the slopes." "It's not a rectangle, the sides are not perpendicular," offered a third student, "I already checked." "Maybe we could make it a rectangle, a rectangle around the whole thing. Then we could see if we can take away the parts that aren't there." That proved to be a fruitful approach. In this classroom, discussion and communication were important in enabling all students to understand the problem.

What is the role of the teacher in supporting communication at grades 9–12?

The high school teacher plays many roles in supporting communication, depending on the context. Varied goals regarding oral communication include creating a climate in which all students feel it is safe to venture comments and conjectures, helping students clarify their statements, helping them refine ideas, and orchestrating the conversations as needed so that the appropriate level of discourse and mathematical argumentation is maintained.

Early in a mathematics course, or when a new topic is being introduced, teachers can open a conversation with a request for information. At the beginning of a tenth grade unit on circles, for example, a teacher collected the following responses to the request, "Tell me everything you know about circles."

Juan: "It's made up of a series of arcs that are all connected."
Monique: "A circle is a shape that has no points."
Louis: "The outside of the circle is called the circumference."
Ramona: "A circle is round."
William: "A circle measures 360 degrees."
Eric: "Radius is half the distance of the diameter of the circle."
Cathy: "1/2 = 180 degrees."
Rebecca: "$A = \pi r^2$ and $\pi = 3.1415927$"
Michael: "It has two sides (inside and outside)."

> Charles: "The center is the radius."

The comments provided evidence of different students' understanding, which varied substantially. This exercise is a useful part of ongoing classroom assessment. But, in this example, the teacher took the discussion further. The next day, the teacher asked the students to agree or disagree with each of the statements, and to justify why they took the stance they did. In this way, the students' knowledge became a starting point for classroom conversations. Also, the idea that one has to have reasons for mathematical opinions was established, and respectful dialogue was begun.

> In an eleventh grade class, students were given the problem "What can you say about the sum of the first n odd numbers? Justify your answer." Most students, having had experience with patterns in earlier grades, tried a few sums and observed that each time the result was a square.
>
> Which one?
>
> The nth square.
>
> Can someone give me a mathematical sentence that says what you found?
>
> The sum is the nth square.
>
> Is there enough information in that sentence to know what you've done? Can you be more precise?
>
> The sum of the first n odd numbers is the nth square.
>
> OK, that's a good sentence in English. Can you help me translate that into mathematical symbols?

With some give and take, the class produced the equation

$$1 + 3 + \ldots + (2n - 1) = n^2$$

and the teacher asked students if they thought they could prove the statement. One student said it could be proved by mathematical induction. The teacher asked a few students who were familiar with induction arguments to work with partners on developing a proof. They did, and when students completed the proof, the teacher made notes for the group on their product. While the small group was working, the teacher said to the class:

> I want to go back to our original statement. You said the sum is a square. Can you draw me a picture? The teacher left that as something for the students to think about, and students came in the next day to discuss how the pictures they produced were related to their conjecture. They also went on to explore related conjectures about the sum of the first n even numbers and the sum of the first n integers.

While there are many student contributions to the interactions, the teacher shapes them in the direction of making a compelling mathematical argument. The mathematical standards for verbal and written exposition—which distinguish the discussion of this problem from a discussion in grades 3-5, where students might observe the pattern but did not have the symbolic tools to pursue it and its generalizations algebraically—are made clear. Moreover, the students are left to ponder a situation that gives rise to a nice set of connections and ultimately a pictorial proof of a collection of algebraic identities. Thus, the role of the teacher involves bringing students into the conversation, helping them clarify what they are saying and symbolize it appropriately, and structuring situations where they have the potential to make interesting connections.

So far, we have focused on the teacher's role in fostering productive classroom interactions and on helping students make the translation from verbal arguments into written ones. Teachers also can encourage students to communicate effectively using written mathematics. For example, problems that require explanations can be assigned regularly, and the class can discuss the adequacy of those explanations. Ample opportunities for students to sharpen their skills at writing mathematically are provided by exercises of various types, such as:

- Planning expenditures for a proposed class trip and writing a letter to the principal explaining how the money will be spent;
- Being hired as a consultant help a business choose between two options (e.g., which taxicab company is the better one to use or which telephone plan is a better buy) and writing a memo justifying the recommendation; and
- Being asked to make a design plan (e.g., a plan for building a rectangular pet house out of a sheet of plywood, so that the volume is reasonably large) and to justify that plan.

Using writing as a means of reflecting on and solidifying what one knows is also valuable. Several kinds of exercises can serve this function. For example, teachers can ask students to write down what they have learned about a particular topic, or put together a study guide for a student who was absent and needs to know what is important about the topic. Another approach is to ask students who have done a major project or worked on a substantial problem over time to compare some of their early and later work and explain how the later work reflects greater understanding. In these ways, teachers can help students develop the skills in mathematical communication that will serve them well both inside and outside of the classroom. In turn, using these skills will help students develop deeper understandings of the mathematical ideas about which they speak and write.

Standard 9: Connections

Mathematics instructional programs should emphasize connections to foster understanding of mathematics so that all students—

- **recognize and use connections among different mathematical ideas;**
- **understand how mathematical ideas build on one another to produce a coherent whole;**
- **recognize, use, and learn about mathematics in contexts outside of mathematics.**

Elaboration: Grades 9–12

In grades 9–12, students should have many opportunities to become aware of the connections among various mathematical topics. They should build upon their previous mathematical understandings while learning new concepts. They also should see rich connections among mathematics, science, and other disciplines and should apply mathematical understanding and computing technology to solve realistic problems.

What is distinctive about mathematical connections in grades 9–12?

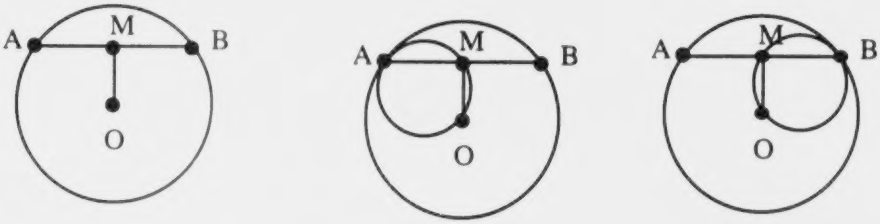
Connections among various content areas help students to see mathematics as an integrated whole and to understand results in different ways. Connections with other academic disciplines, especially the sciences and the social sciences, should become increasingly realistic as students' mathematical knowledge—in particular, of functions and statistics—increases and as their ability to use more mathematical representations is enhanced, both with and without electronic technology.

Students' ideas about connections should mature significantly during high school. In the first place, students should be able to view mathematical structures as objects and systems. Secondly, they should have the flexibility when solving problems to take advantage of the connections among different topics in order to select an appropriate method for solving the problem at hand. Often, using representations from different domains of mathematics can make a rather forbidding problem more accessible. For example, some interesting results in synthetic geometry can be justified with relatively straightforward algebraic computations on vectors. The midpoint connection theorem that the line joining the midpoints of two sides of a triangle is parallel to and half the length of the third side can be justified using the generalization of the following identity which holds for two-dimensional vectors:

$$\tfrac{1}{2}(B-A)=\tfrac{1}{2}B-\tfrac{1}{2}A\,.$$

Students in grades 9–12 should build their capacity to link mathematical ideas. As their mathematical repertoire expands, students can develop a deeper understanding of how more than one approach to the same problem can lead to equivalent results, even though the approaches might look quite different. Students can use insights gained in one context to verify conjectures generated in another and, through linking mathematical ideas, they can develop robust understandings of problem settings.

The connections between algebraic and geometric ideas grow in prominence during grades 9–12. In a dynamic geometry environment, for example, students were asked to explore the relationship between the length of a chord of a circle of radius 1 and the distance from the center of the circle to the chord (see figure 7.22(a)).



(a) (b) (c)

Figure 7.22. Chord problem.

The expectation was that students would gather data for several chords, reason about the situation, and generate a rule that would seem to fit the data. The data they collected are presented below:

Length of $\overline{MO}$	0.81	0.99	0.97	0.94	0.84	0.72	0.63	0.49
Length of $\overline{AB}$	1.17	0.30	0.48	0.72	1.10	1.39	1.57	1.75

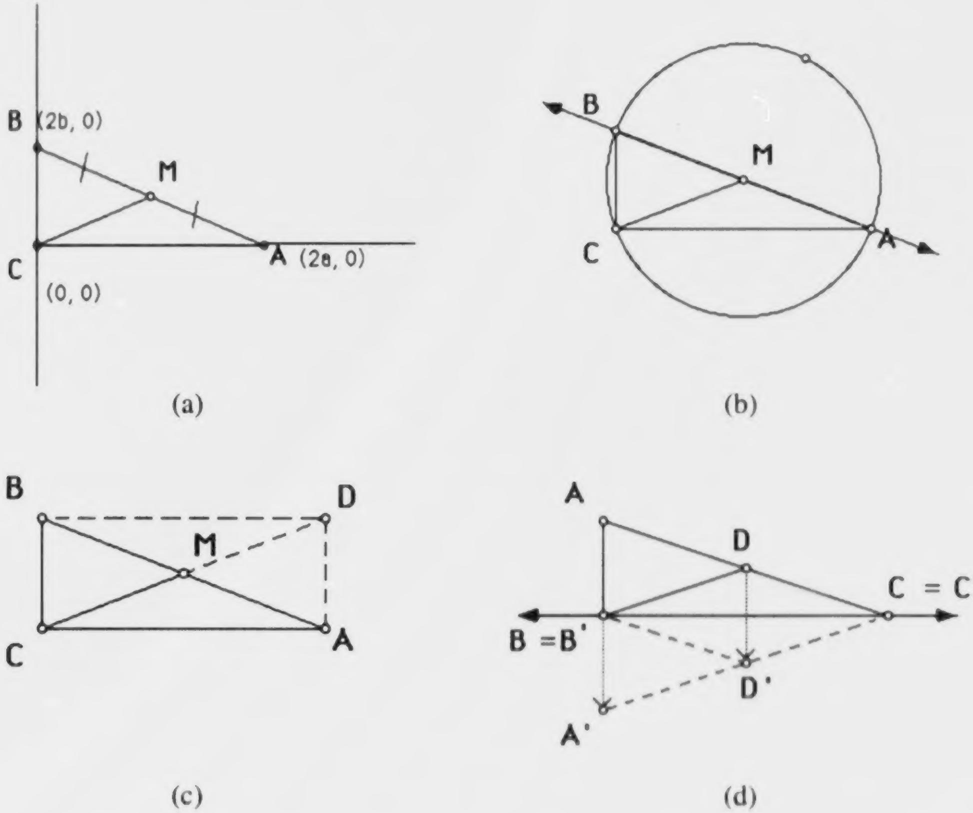
An application of the Pythagorean Theorem revealed that $MO = \dfrac{\sqrt{4 - AB^2}}{2}$, which can be re-expressed as $\dfrac{AB^2}{4} + \dfrac{MO^2}{1} = 1$. This is the canonical form for an ellipse. Students might be asked to graph the ellipse and explore its relationship to the original problem.

A second example highlights the connections among what would appear to be very different representations and approaches to a mathematical problem. Several solutions approaches shown here come from Lewes, O'Keefe, and Goldenberg 1992). Students in this case were asked to consider the following theorem:

> The midpoint of the hypotenuse of a right triangle is equidistant from the three vertices of the triangle.

There are several approaches to proving this theorem (figure 7.23). Each draws on different mathematical connections and features different aspects of the relationship. A reasonable selection of coordinates makes an analytic geometry proof easily accessible. The selection of coordinates shown in figure 7.23(a) yields a common distance of $\sqrt{a^2+b^2}$, possibly reminding the student of connections to the Pythagorean Theorem. Connecting the knowledge that any triangle can be inscribed in a circle with the fact that the measure of an inscribed angle is half the measure of the intercepted arc yields a second proof (see figure

7.23(b)). A third justification derives from constructing a rectangle using the three vertices of the triangle (see figure 7.23 (c)) and relies on students reasoning about the properties of the diagonals of a rectangle. A fourth solution, arising from transformational geometry, is based on recognizing and using the invariants of plane figures under reflection in a plane (see figure 7.23(d)). In the transformational geometry proof, students would note that since reflections preserve lengths, D′ is the midpoint of A′C′ and B is the midpoint of AA′. Applying the midpoint connection theorem to triangle A′CA shows that BD = A′D′ = AD, as desired. The transformational geometry proof highlights a variety of other relationships between parts of the triangle and its image under reflection.



(a) (b)

(c) (d)

Figure 7.23. Various proofs of midpoint to hypotenuse theorem.

Dynamic geometry tools can help students can appreciate the generality of their reasoning. They also provide visual models for thinking about relationships. The relationship between the midpoint of the hypotenuse and its length is illustrated in figure 7.24. Part (a) shows a set of frames from a dynamic geometry animation of the relationship between lengths—the set of right triangles has a fixed hypotenuse but the leg lengths vary.

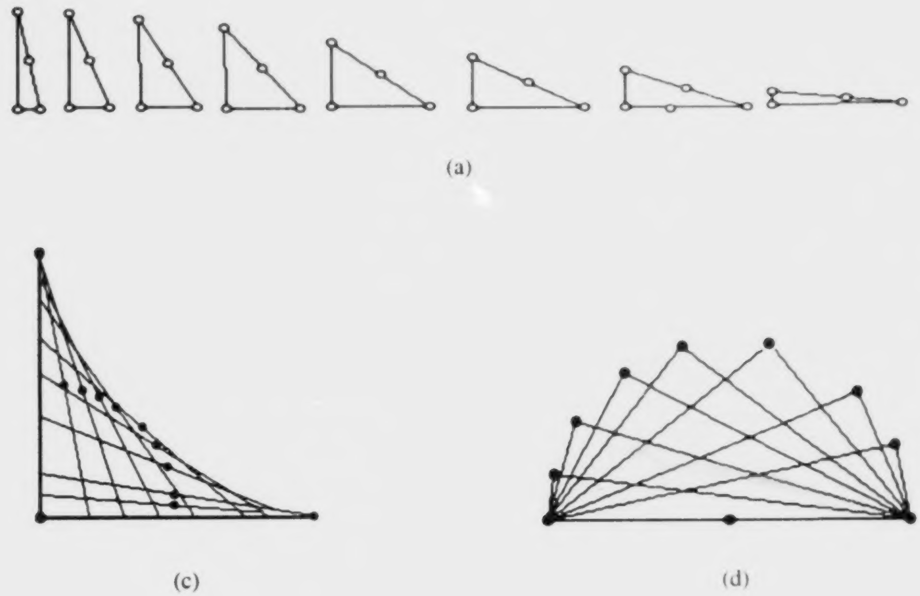


Figure 7.24. Dynamic representations of right triangles.

Part (b) illustrates the superposition of the triangles so that right angles coincide; part (c) shows the superposition of the triangles so that hypotenuses coincide. Both figures (b) and (c) suggest a circular pattern for the midpoints. Part (b) suggests that the midpoint of the hypotenuse for each of the triangles is equidistant from the right angle vertex, and, through a limit argument might suggest that the midpoint is equidistant from all three vertices. Part (c) suggests that, for each of the triangles, the midpoint of the hypotenuse is equidistant from the three triangle vertices.

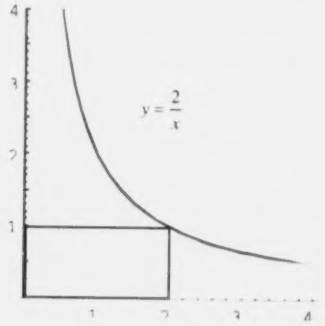
Connections among various content areas allow students to view different areas of mathematics from other mathematical perspectives, thereby enriching their understanding of both the area being studied and the area used for an alternate representation. For example, the binomial theorem can link algebra and numbers—note that $2^4 = 16 = (1+1)^4 = 1+4+6+4+1$. The realization that fourth degree polynomials usually have four roots can lead students to wonder where the missing two fourth roots of 16 (besides +2 and –2) might be. DeMoivre's Theorem finds the other two roots on the imaginary axis in the complex plane, but still on the circle $|z| = 2$, thus giving insight into the geometry of complex numbers.

In grades 9–12, students' capacity and repertoire for linking mathematical ideas to ideas in other disciplines increases. For example, familiarity with the slow-growing logarithm function helps provide a useful scale for expressing the strength of sounds or earthquakes by using reasonable numbers rather than numbers that quickly become too large or too small to be easily understood. An earthquake of magnitude 4 on the Richter scale is

10,000 (10^4) times as intense as one of magnitude 1 (10). Students can combine their knowledge of proportionality with their increasingly richer understanding of the graphical representations of functions to interpret distance vs. time and velocity graphs. In grades 9–12, students should use their study of exponential functions to understand compound interest from the points of view of both a saver and a borrower and appreciate the dramatic effects of exponential growth over time. The same mathematics can be linked to biology, thus helping students to understand the significance of the recent dramatic growth in the world's human population, as well as the danger of rapid transmission of communicable diseases. Connections between mathematics and the sciences often become apparent when students engage in the modeling of physical phenomena, such as finding the speed of light in water, determining proper doses of medicine, or optimizing locations of fire stations in forests.

Previously, problems such as maximum-minimum problems or others that call for the analysis of mathematical models were deferred until students took calculus courses because the analytic techniques of calculus were needed. Now, however, students can engage in the modeling long before calculus. If the result is an analytic expression, they can determine the properties of the resulting graphs with the help of graphing utilities and symbolic manipulation technologies. Consider the following example, that connects geometry, functions, and algebra.

> a. Determine the maximum area enclosed by a rectangle, two of whose sides are on the coordinate axes, one vertex is at the origin, and the opposite vertex is on $y = 2/x$. Is your answer exact or approximate?
>
> b. Determine the maximum perimeter. Determine the minimum perimeter. Are your answers exact or approximate?



The solution to part (a) is not initially intuitive. Function concepts can be used to develop an equation for the area and perimeter. Students should be asked to give verbal interpretation of the results to part (b). Dynamic graphing programs coupled with linked symbolic tools also can help students explore the properties of functions as their parameters vary.

What is the role of the teacher in helping students make mathematical connections?

To enhance students' capacities for seeing and using mathematical connections involves engaging them in problems and situations carefully chosen to reveal connections within mathematics and of mathematics to other disciplines. This requires particular initiative on the part of the teacher when, as is often the case, available curriculum materials separate the study of algebra from the study of geometry from the study of statistics. Even if teachers work with integrated curricula, however, they will still need to develop expertise in making mathematical connections and in helping students develop their own capacity for drawing out such connections.

As the grade level increases, the depth and breadth of the mathematics available to teachers and students for making connections also increases. At the grade 9–12 level, this means that teachers need to understand an ever-increasing amount of mathematics. Teachers must understand this mathematics deeply and develop a well-tuned ability to recognize the ways in which mathematics can describe situations from other disciplines. They must develop the inclination and capability for making connections among ideas across a gamut of mathematical areas. The teachers also needs to be alert for connections that the students may make on their own in class and then support those observations and help all students see their value.

A tenth grade student who had studied geometry was studying the absolute value function, $f(x) = |x|$. She observed the graph of the function and said, "This reminds me of studying angles in geometry last year. Is this a way to get a formula for any angle?" This student was naturally looking for a connection to earlier work. The teacher should take full advantage of such opportunities. It is not enough for teachers alone to be skilled at finding and making mathematical connections in the classroom. Teachers should introduce students to fertile mathematical settings in ways that make them curious about the mathematics and about the mathematical structures that underlie the connections. They should entice and enable students to explore those mathematical structures, especially when unexpected opportunities arise. Moreover, teachers need to help students see the ways in which mathematical structure helps them describe their worlds. This is especially true at the 9–12 level because the material becomes increasingly abstract and general.

In summary, teachers need to help their students not only to wonder about connections but also to have the mathematical tools to pursue them. Teachers should develop classroom atmospheres in which seeking and explaining connections is an expected part of the interplay between the students and the mathematics.

Standard 10: Representation

Mathematics instructional programs should emphasize mathematical representations to foster understanding of mathematics so that all students—

- **create and use representations to organize, record, and communicate mathematical ideas;**
- **develop a repertoire of mathematical representations that can be used purposefully, flexibly, and appropriately;**
- **use representations to model and interpret physical, social, and mathematical phenomena.**

Elaboration: Grades 9–12

Students understand mathematical ideas and relationships and communicate them to others through the representations they create or adopt. As they emerge from middle school mathematics programs, students will have encountered algebraic, graphic, and numeric representations for many mathematical objects and their properties. At the high school level, there is a greater number and type of mathematical objects and relationships that students should be expected to represent and interpret. New representational forms offer additional ways of understanding and communicating about content and building connections. New models and methods for describing and interpreting phenomena that occur in the world are available to students. As they progress through high school, students must continually develop their abilities to meaningfully interpret and use conventional, and increasingly abstract, forms of representation fluently and flexibly.

What is distinctive about representation in grades 9–12?

In grades 9–12, students' knowledge and use of representations should expand along at least three dimensions.

First, students should continue to develop their understandings of representations that were introduced in earlier grades and continue to develop the connections between them. Students also should be gaining familiarity with new representations, including matrices, tree diagrams, and recursive and parametric forms of functions. For example, the recursive form, $f(n) = f(n - 1) + 2$, where $f(1) = 1$, defines the same function as $f(n) = 2n - 1$ where n is a natural number . There are situations when a recursive formula is more helpful, or even necessary, in describing a particular relationship.

Once students can use representations as tools for characterizing patterns, they can look more deeply into the structure of the patterns. Consider a situation, for example, in which one object grows continuously so that its weight doubles every hour and a second object that grows $5n$ grams in the 7th hour. If the two objects start with an initial weight of 2 grams, at what time will their weights be equal? Creating a table of values, such as the one shown below can help students develop an initial understanding of the situation.

Time in hours	0	1	2	3	4	5	6	7	8
Weight of 1st object	2	4	8	16	32	64	128	256	512
Weight of 2nd object	2	7	17	32	52	77	107	142	182

Students might compare these two situations using graphs:

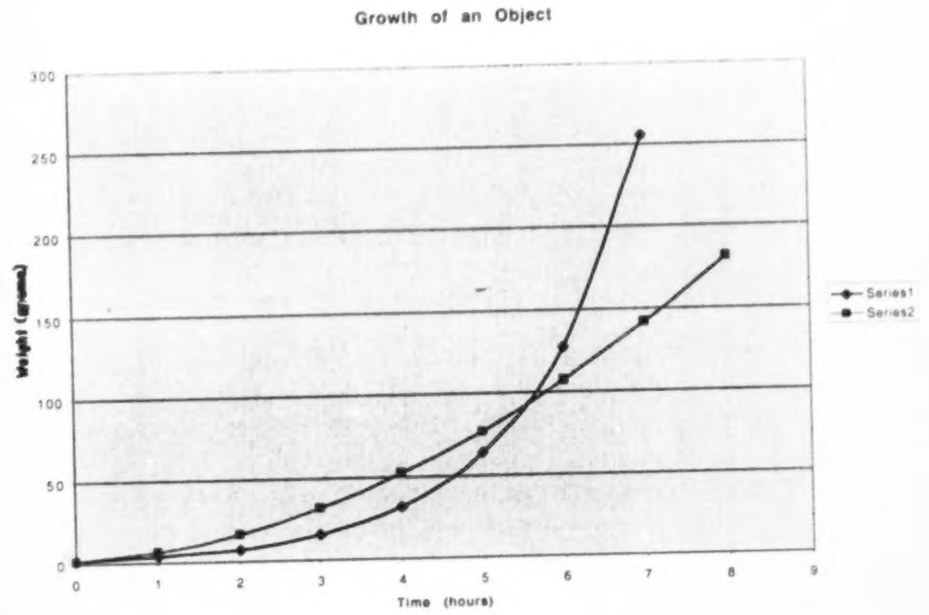


Figure 7.25. Growth of an object.

From the graphs, it appears that the two situations could be modeled by functions that would cross exactly twice. Students' knowledge about the underlying classes of functions (exponential and quadratic) might provide support for this visual conjecture. However, it takes manipulations of the symbolic forms (ultimately the derivative of each) to provide a mathematically sound argument that they, indeed, do not intersect at any points other than $t = 0$ and somewhere between $t = 5$ and $t = 6$.

Second, students can use representations at more abstract levels. In elementary school, students often use representations to reason about contexts in which they can directly perceive the objects and the actions performed on them. During the middle school years, students increasingly create and use mathematical representations for objects that they do not necessarily perceive directly, such as the location of a train, plane, or bicycle and the action of moving from one location to another over time. In high school, students should learn to use representations as a primary means by which to express and understand concepts that are more abstract. They should learn to use representations to elicit common mathematical structure across contexts. For example, the same linear function may represent distance traveled as a function of time, or height of a stack of cups as a function of the number of cups. The sum of the first *n* consecutive odd natural numbers, the area of a square field, and the distance traveled by a vehicle that starts at rest and accelerates at a constant rate can all be represented by a function of the form $f(x) = ax^2$. The fact that different situations can be represented by the same type of equation implies that they all are alike in some fundamental mathematical way.

Third, high school students should be able to model phenomena taken from a wider range of contexts—including physical and social environments—by identifying essential elements of the context and by devising representations that capture mathematical relationships among those elements. By working in a range of contexts, high school students will encounter exponential, quadratic, periodic, and logistical behavior. Fundamental understanding of periodic or cyclic phenomena often are best modeled by trigonometric functions: population growth usually follows an exponential model, and predator/prey situations often grow logistically. In high school, students' mathematical knowledge of these classes of functions and their representations can be applied. Traditional analyses can be enhanced with graphical analysis, regression tools, and modeling software.

In addition to expanding their representational repertoire in these three dimensions, high school students also should recognize the advantages and the limitations afforded by different types of representations. They should be able to move flexibly and effectively among the various representations of an object, action, or concept to gain and communicate important information about the situation.

Electronic technologies provide access to problems and methods that until recently were difficult to explore meaningfully in high school. Iterative numerical methods, for example, can be used to develop an intuitive concept of limit and its applications. Asymptotic behavior of functions is more easily understood graphically, as are the effects of transformations on functions. These enriched understandings give students access to models that can be used to analyze realistic and more interesting situations. In this way, students will have an enhanced ability to solve problems, use whatever tools seem appropriate, and communicate their understandings effectively.

What is the role of the teacher in helping students learn about representation at grades 9–12?

Teachers need to pay attention to the representations developed by their students and ask questions as needed to be sure that they understand how students are conceptualizing the mathematics. Students often have very effective ways of thinking about problems and relationships. A major part of ongoing classroom assessment for teachers is working to understand student thinking, assessing its key elements, and reflecting on how they can help students develop it further. As teachers listen carefully to understand student's initial ideas, part of their role is to help the student connect personal images to more conventional representations without losing the sense of understanding along the way. To illustrate this point, consider a familiar type of problem:

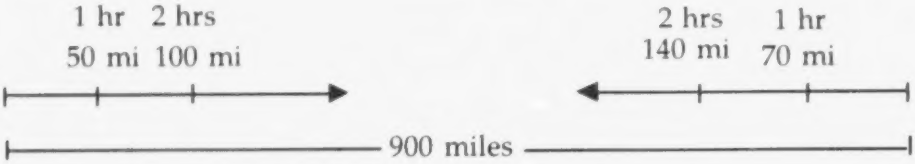
> Two trains are heading toward one another on parallel tracks. One train is traveling 50 miles per hour and the other 70 miles per hour. If they start 900 miles apart, how long will it take them to meet?

The teacher, who might be inclined to work with equations, needs to be open to solutions that students might generate and help students work effectively from their representations. In this example, a student generated the following table:

	hours 1	2	3	4	5	6	7	8	9
Train 1	50	100	150	200	250	300	350	400	450
Train 2	70	140	210	280	350	420	490	560	
						720	840	960	

They meet somewhere between 7 and 8 hours.
What happens at 7½ hours? Train 1 will
be at 375 mi, and train 2 at 490 + 35 = 525.
375 + 525 = 900. Lucky guess!

Another student began with this diagram:



and added in additional hours with the goal of finding a sum reaching 900. Another used the equation $50t + 70t = 900$. The teacher can lead this class in a discussion of differences in understandability, efficiency, and generality of these various approaches.

Too often, when students are confronted with the formal symbols of algebra or of geometric reasoning, they become disconnected from the content and the learning and feel that the representation has become an end in itself instead of a device for understanding and communicating. Linking conventional representational systems to students'

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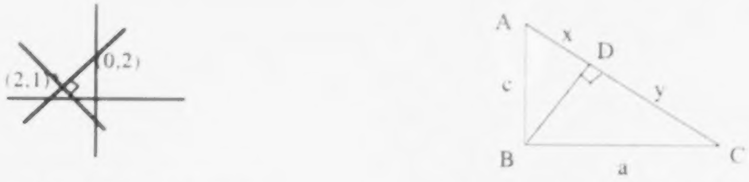
own understanding is the most likely way of avoiding this pitfall. In addition, it is important that teachers not necessarily interpret students' different conceptualizations and representations as wrong or indicative of a lack of understanding but instead work to understand student thinking. This is no easy task. It requires knowledge, ongoing assessment, and sensitivity on the part of teachers

One strategy used effectively by teachers is to offer students a representation such as a simple graph or a geometric figure and to ask students to write and share everything they think they know about the diagram, as in figure 7.26.



Tell me everything you know about the graph shown here.

Tell me everything you know about the triangle shown here.

Figure 7.26. Open ended description tasks.

When students are given tasks like those shown in figure 7.26 they probe their own representational understanding. The teacher has the opportunity to assess where to begin the lesson and what level of assumed knowledge is appropriate for this class and for individual students within the class. Learning occurs as students think deeply about their own associations and listen carefully to what others know. The teacher can build on this common set of understandings to develop the lesson and to link to more formal statements of the properties of each figure.

Students need to understand the efficiency and eloquence of mathematical form and abstraction and be able to use it to their advantage in pure mathematics and modeling environments .Classroom environments that are supportive of their own thinking, and that also move them toward more conventional representational schemes, support such understanding and use.

Chapter 8
Issues and Implications for Principles and Standards

#10

Reader Reaction

The development of Chapter 8 will be based in part on response and feedback gathered during the 1998-99 school year. Its purpose will be to highlight issues in implementing the ideas of *Principles and Standards* and to provide policy guidance. Please contribute to this process by responding to the following questions:

- **For each of the following levels of mathematics instructional programs, what *issues* need additional discussion?**
  - **(a) Classroom level**
  - **(b) School level**
  - **(c) State or province level**
  - **(d) National level**
- **What *policy guidance* might be offered in each of the following areas?**
  - **(a) Instructional materials development**
  - **(b) Mathematics teacher preparation**
  - **(c) Teacher professional development**
  - **(d) School administration**
  - **(e) Higher education**

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